TRANSPORT PROCESSES IN PLASMAS

1. Classical transport theory

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Preface

The problem of the transport of matter, momentum and energy in a plasma submitted to temperature, pressure and velocity gradients combined with external electric and magnetic fields is, without any doubt, one of the most crucial aspects of plasma physics. In a nutshell, it can be said that *the object of transport theory is the response of a system (here: a plasma) to a sustained external constraint*. Let us make this statement more explicit.

Consider a material system which is in some kind of stationary state (it may be a true thermodynamic equilibrium state, or some more general kind of quasi-steady state, examples of which will be met later). At some time, a "*thermodynamic force*" is applied, which disturbs the initial equilibrium. This force may be an external force field, or it may be a spatial inhomogeneity (i.e., a gradient) of density, velocity or temperature. The system *responds* to this stimulus: various kinds of motion set in. There may be an overall material motion of the system, but also more subtle flows of momentum, energy or electric charge, which are not necessarily accompanied by a global motion of the matter.

The general tendency of the system is to react in such a way as to recover its initial steady state (*if* the latter is *stable*) when the driving forces are removed. If, however, the forces are sustained, many differrent things can happen: the system evolves now under an *external constraint*, opposing its return to equilibrium.

If the forces are maintained at a steady level and are not too strong, the system *may* (or may not!) reach, after some time, a new, *non-equilibrium steady state*, in which its state variables (such as the density and the temperature) retain constant values in time, but in which there exist non-vanishing, *constant fluxes* of matter, momentum, energy and/or electric charge. If, however, the constraint is too strong, or if it varies in time, the response of the system will be much more complex.

In any case, the result of the application of a thermodynamic force will be a redistribution of matter, momentum, energy and charge throughout the system, hence the appearance of more or less violent fluxes of these quantities. We entered the realm of *non-equilibrium physics*, a field that is in full expansion in the present days.

The importance of transport theory to plasma physics is enormous. It can be briefly illustrated by the following examples.

The problem of *Controlled Thermonuclear Fusion* is based on the possibility of confining a plasma inside a magnetic field configuration for a time long enough to permit the start-up of the nuclear fusion reactions and the collection of the energy produced by these reactions. But such "magnetic bottles" are

never tight: there are unavoidable "leaks" which may be fatal to the confinement. The control of these leaks implies a perfect knowledge and mastership of the mechanisms of transport of matter and energy from the region of production or deposition (i.e., in the core of the plasma) towards the walls, in the presence of a strong confining magnetic field of complex toroidal geometry.

Astrophysics and Geophysics is another important field of applications of plasma physics, as more than 90% of the matter in the universe is in the plasma state. Here again, the problem of transport is crucial in understanding a variety of complex phenomena. To quote just a few examples, the problem of the solar phenomena (flares, protuberances, spots,...) requires a deep study of magnetohydrodynamics in conditions where dissipation, hence transport, is quite important. The fascinating processes of magnetic reconnection are basic in the understanding of the phenomena (such as magnetic storms) occurring in the magnetospheres surrounding the earth as well as other planets, each with its own specificity. A fundamental question in the theory of reconnection is the mechanism of "anomalous transport" which produces the necessary dissipation. The latter, in absence of sufficiently frequent interparticle collisions, must arise from subtle collective electromagnetic phenomena, including plasma instabilities and turbulent processes.

Finally, we may briefly mention the more daily applications of plasma physics to such phenomena as electrical discharges in gases, arcs,..., where the problems of electrical conductivity and thermal conductivity are again fundamental.

Besides these motivations arising from the applications, we may stress the interest of transport theory for the theoretical physicist in the general framework of *non-equilibrium statistical physics, kinetic theory* and *non-equilibrium thermodynamics*. This problem was already of great historical importance since the times of the founders of the kinetic theory of gases: Maxwell, Boltzmann, Chapman, Enskog, Cowling. Since the relatively recent development of plasma physics, an entirely new dimension appeared in transport theory. Unlike neutral gases, plasmas consist of electrically charged particles interacting through long-range forces: this in itself induces a vast variety of new phenomena (especially collective behaviour of the particles). Moreover, because of their electric charge, these particles are affected by external magnetic and electric fields. This feature opens up a new level of interaction between the system and the experimentalist. The latter may control the transport not only by imposing temperature and pressure gradients, but also electromagnetic fields of arbitrarily complex and subtle shape and time dependence. In turn, the system may react to these external fields in ways which are often not predicted by straightforward intuition: it may develop peculiar drift motions, or instabilities that may lead to the breaking of the initial topology of the

magnetic fields, and in the last stages, the motion (as well as the internal electromagnetic fields) may become completely chaotic and turbulent. All these features strongly influence the transport mechanisms. As a result, the transport theory of plasmas is an incommensurably richer field than the classical transport theory of neutral gases.

I have been "in love" with the statistical physics of plasmas over my whole scientific life. Having studied for years the problems related to the "kinetic stage", i.e. the derivation of irreversible kinetic equations from the reversible equations of classical and quantum mechanics. I wanted to complete the programme of statistical physics by going over to the "hydrodynamic stage", in which the microscopic, molecular information is transferred to the observable, macroscopic level. Curiously enough, this stage has been somewhat neglected by statistical physicists in recent years. This is the more regrettable because, particularly in the field of plasma physics, the "hydrodynamic stage" leads to the plasma transport theory with all its fascinating aspects.

For many years I was surprised by the absence of a specific monograph on this subject. At present, a physicist wanting to enter this field has to begin his study with five "classical" review papers:

– S.I. Braginskii, 1965, in: Reviews of Plasma Physics, Vol. 1, ed. M.N. Lenotovich (Consultants Bureau, New York) [for the classical theory].

– A.A. Galeev and R.Z. Sagdeev, 1979, Reviews of Plasma Physics, vol. 7.

– F.L. Hinton and R.D. Hazeltine, 1976, Reviews of Modern Physics.

– S.P. Hirshman and D.J. Sigmar, 1981, Nuclear Fusion [the latter three references concern the neoclassical theory].

– P.C. Liewer, 1985, Nuclear Fusion [for the anomalous transport].

He then must start the search for specific research papers, scattered in a large number of journals, conference proceedings or laboratory reports. He will have difficulties in the fact that these papers (including the review papers!) do not try to present a global formulation within a general framework of plasma transport, but rather present a special chapter from a specific point of view.

The existing general textbooks or monographs on plasma physics usually contain only a very small chapter devoted to plasma transport: it is presented in a very elementary and non-rigorous way.

I therefore decided to try to fill this gap and started working on the project at the beginning of 1981. I soon began to understand the reason of this gap in the literature. I realized that, if the subject were to be treated in the spirit I had chosen, it would be absolutely impossible to cover the whole matter in a single volume, even after the elimination of less essential aspects.

On the other hand, I realized that this matter could be divided in two volumes, which could be conceived as self-contained entities. The first one would include the so-called *classical and neoclassical transport theories* and the

second one the *anomalous transport theories*. In the first volume, one studies the transport mechanisms explained by the properties of *individual particles*, which are the actors of the interactions among themselves and with the external electromagnetic fields. In the second volume one would deal with the *collective transport mechanisms* mentioned above. At the end of the work on the first volume, it appeared that even this would give rise to a too thick book, whose use would be unhandy: it was therefore decided to split it into two parts. The numbering of the chapter remained, however, continuous, in order to stress the intrinsic unity of the work. Thus, Part I is devoted to the *classical transport theory* and contains chapters 1–7, and Part II devoted to the *neoclassical transport theory*, contains chapters 8–19. The work on the second volume, devoted to the anomalous transport is, at present, in a preliminary stage and will not be published before several years.

I now wish to comment on the general spirit of this work. My objective was to write a self-contained book, which the reader could study without having to resort to side-references at every step. I therefore chose to present plasma transport theory as a discipline strongly embedded in the framework of theoretical physics. By the latter statement, I mean not only "plasma physics" in a restricted sense, but more generally: Hamiltonian mechanics, statistical physics, thermodynamics, hydrodynamics, electrodynamics. Plasma transport theory precisely realizes a synthesis of all these fields: it can be characterized as a truly *interdisciplinary activity*. This explains the presence of several chapters in which the important concepts of these "peripheral" fields are introduced, briefly but completely.

In making the unavoidable selections of the material, I decided to focus on relatively simple situations. Thus, the plasma model used throughout the book is a simple, fully ionized plasma consisting of electrons and a single species of ions. Within this framework, the presentation is as exhaustive as possible: the calculations are given in great detail) much greater than in the published papers) in order to make its study as easy as possible. The assumptions are discussed and every crucial step in the argument is commented both in its physical and its mathematical aspects.

I developed a formalism which allows the presentation of all the features of classical and neoclassical transport in a unified way, by using the same tools, concepts, methods and notations throughout the book. The basic tool of this formalism is the Hermitian moment method of solution of the kinetic equations (chs. 4, 11, 12). Although inspired from the classic work of GRAD, it goes well beyond its initial formulation, making it applicable to the whole range of problems under consideration here.

The final results are given in the form of completely explicit and simple *analytical formulae* (a new feature!). These formulae enable one to calculate numerical values of the transport coefficients with excellent precision on a

pocket calculator. A large number of tables, figures and graphs are presented, a feature usually absent in the transport literature. It is extremely helpful (in my opinion) for the general intuition of the problem to have a pictorial view of the phenomena. Moreover, many of these graphs allow a simple and rapid reading of the numerical values of the transport coefficients.

A monograph like this one implies a large amount of compilation of a vast amount of literature. I should like to stress, however, that the presentation is never left at the primitive stage of a compilation, or even of a critical review paper. I endeavoured to rethink all the known results and reformulate them in the framework of a unified theory. Every single calculation in the book was done and redone several times. As a result, the text went through several successive versions before its finalization in its present form.

The level of the mathematics is rather unsophisticated. Most of the problems treated here appeal only to linear algebra and to elementary aspects of differential equations. This should make the book accessible to a wide audience of experimental physicists, as well as to good undergraduate students. The only less usual mathematical tools appearing here are those related to the toroidal geometry: all the necessary concepts are discussed in detail in the text and in the appendices.

The prerequisites are just the basic principles of statistical physics, kinetic theory, electromagnetism and a feeling of plasma physics, as can be expected from any physicist or engineer.

In conclusion, my main purpose in writing this book was a comprehensive and unified presentation of the transport theory in plasmas. The reader will judge the degree of success in the achievement of this endeavour.

PART I Classical transport theory

... πάντα ρεί ...
(... *all things are flowing* ...)

Heraclitus?
quoted by Aristotle.

Contents

Introduction

Part I of this work is devoted to the *classical transport theory* in plasmas. Th meaning given to (or implied for) this term in the literature is somewhat imprecise and largely conventional. In the present work we choose a restricted, operational definition for this title.

The classical theory covers the transport phenomena in a plasma, considered as a collection of *charged particles interacting through binary collisions*, in the presence of *straight, homogeneous and stationary magnetic and electric fields*.

For denser plasmas, three- or four-body collisions can also be taken into account in this framework. But the effects related to truly collective interactions (through "waves", "vortices", "clumps", etc.), which exist in a plasma as a result of the long-range Coulomb forces, fall outside of the scope of the classical theory and are the object of anomalous transport theory. On the other hand, the magnetic field inhomogeneity and curvature produces quite specific effects on the transport: these are studied under the heading of the neoclassical theory.

Even in this classical field, developed since many years, in which one could think that everything was solved a long time ago, it is amazing to note the enormous disparities in the presentation of the results by the various authors, as a result of (often unstated) differences in the starting assumptions. I therefore tried to present (see chapter 5) a complete, self-consistent and self-contained view of the classical plasma transport theory. This presentation turns out to be somewhat different from the usual ones (such as the universally quoted Braginskii formulation), for reasons discussed in detail in the text. The advantage of the present form is its natural insertion into the general framework of non-equilibrium thermodynamics, in particular in connection with the *entropy production*, a concept that allows a deep understanding of the structure of the transport theory. This subject is, unfortunately, not often treated in detail in the plasma physics literature. It also leads to a natural continuation into the domain of the neoclassical theory, hence to a unified transport theory.

Many results or methods used here are new. Let me quote as an example the presentation of the motion of charged particles in chapter 1. Unlike the usual textbook presentations (based on not too well defined time averaging methods) I chose to make use here of the elegant modern methods of

Hamiltonian mechanics which allow a clear, precise and simple derivation of the guiding centre motion. Such a presentation is a "première", never yet used in existing textbooks. Another example is the systematic use of *irreducible Hermite polynomials* as a basis for the expansions in velocity space. It appears that these functions lead to much more transparent and compact formulae than the Laguerre–Sonine polynomials, widely used in kinetic theory. As a final example, I wish to underscore the application of non-equilibrium thermodynamic methods and the analysis of the entropy production in connection with plasma transport theory.

The general structure of the work reflects the natural progression from the microscopic description of the plasma to its macroscopic picture as a fluid. This is nothing other than the implementation of the programme of statistical mechanics.

Chapter 1, therefore, starts with the study of the motion of an *individual charged particle* in the presence of an electromagnetic field: this is a chapter of "pure" Hamiltonian mechanics.

Chapter 2 introduces the tools of statistical mechanics for the study of *large collections of charged particles*. It culminates in the derivation of a *kinetic equation*, which provides us with the basic tool of transport theory.

In chapter 3, we go over from this intermediate level of molecular description to the macroscopic level, by deriving the hydrodynamic – or, better, the plasmadynamic – balance equations. At this stage we meet with the intrinsic difficulty of macroscopic physics: there are more unknowns than equations! More precisely, the macroscopic dynamical equations have the structure of an infinite hierarchy. This introduces the necessity of constructing a *transport theory*, by which the infinite set of equations can be reduced to a finite, closed set. This can only be done by a detailed analysis of the kinetic equation, under well-defined conditions. The tools for such an analysis are developed in chapter 4.

Chapter 5 is the central chapter of this volume. The *transport equations*, relating the unknown fluxes of matter, momentum, energy and electricity to the hydrodynamic variables (i.e. more precisely, to the thermodynamic forces) are derived and discussed. The domain of validity of these equations defines the realm of *classical transport theory*. It covers the situations where the collisions dominate the evolution: the mean free path of the particles between two collisions is much shorter than the typical macroscopic (hydrodynamic) lengths.

In chapter 6 the results are incorporated into the wider framework of non-equilibrium thermodynamics, by connecting the transport processes to the central concept of entropy production.

Finally, in chapter 7, the results of transport theory are put back into the equations of plasmadynamics, which are now closed and ready for use...

As a final comment, I might try to answer a question which a pragmatic plasma (in particular, fusion plasma) physicist might ask: "Why bother about the classical transport theory, when it is well-known that it does not provide us with the final explanation of experimental facts?". A philosophically-minded scientist, inspired by Popper's views, might even say that classical transport theory has been "falsified" by experiment.

I might answer these objections by first noting that classical transport theory is an example of a complete, self-consistent and fully organized theory. Besides its aesthetic appeal, it provides us with a solid understanding of the basic transport processes.

Next, I would remark that I have a view of the development of physics that is rather different from Popper's (and which I share with many other theoretical physicists, in particular B. d'Espagnat). Science does not progress by successive "falsifications" – and, thus, rejections – of existing theories and their replacement by entirely different ones. The progress rather goes through a succession of gradually larger, but concentric circles. A new theory generalizes the present one, but the latter appears as a particular case of the former and remains perfectly valid within a well-defined domain. Classical mechanics was not "falsified" by the advent of relativity or of quantum mechanics. In the 1980's many physicists work with classical mechanics, and many spectacular new results were obtained in this "old" field.

In the same sense, classical transport theory retains its importance and may still progress in the present days. Its role is not only confined to yielding precise quantitative answers within its domain of validity. Its structure and its tools will become the starting point of all the forthcoming developments and even its detailed results (such as the parallel transport coefficients) will appear as basic building blocks of the more general theories. I endeavoured to prove these statements in Part II of this work, devoted to the neoclassical transport theory.

1

Motion of a charged particle in an electromagnetic field

1.1. Introduction

The whole subject of plasma physics can be reduced, in last analysis, to the understanding of the motion of a set of charged particles in an electromagnetic field. Having said this, we immediately stress the misleading nature of this simple statement. As soon as it is expressed on a quantitative basis, one realizes the overwhelming complexity of the problem.

Consider a *single charged point particle* entering a region of space permeated by an electromagnetic field, produced by some unspecified external sources. When the latter reduces to a *constant electric field* $\boldsymbol{E}$, every freshman knows that the particle will undergo a uniform acceleration in the direction of the field. On the contrary, in the presence of a *constant magnetic field* $\boldsymbol{B}$, the particle will perform a helical motion, wrapping itself around a field line. When the two fields are present simultaneously, the motion is more complex, but is still amenable to exact analysis. The particle no longer remains attached to a single magnetic field line, but rather "*drifts*" through space in a direction perpendicular to both the electric and the magnetic fields.

A constant electromagnetic field is a fiction which can, at best, be realized only as an approximation valid in a restricted region of space and for a limited span of time. Any realistic field is *spatially inhomogeneous* and *nonstationary*. Clearly, even the slightest dependence of the fields on space and time makes the equations of motion *non-linear*. The latter can no longer be solved exactly. Two possibilities offer themselves. As a first choice, the equations can be solved numerically with the help of a computer. This methods is important for specific problems (such as the design of a particular experiment). However, like all numerical methods, its results cannot be used directly for constructing a general theory. A second choice is to develop specific approximation methods, which allow the equations of motion to be solved analytically. (Note that numerical results are often an invaluable guide for the construction of analytical approximations!) Although limited by a set of validity conditions, such methods have the enormous advantage of leading to the construction of a

true theoretical framework in which an extended set of physical situations can be treated in a transparent way.

These complicated problems are still far from exhausting the necessary knowledge requested for plasma physics. Indeed, we now consider the situation obtained by injecting not one, but *several charged particles* into the region occupied by the external electromagnetic field ("several" means about 10^{23} for any sizeable bit of a plasma!). Now we are stepping into an altogether superior realm of complexity. The individual particles are no longer passive objects which are pushed or pulled from one side to the other by a set of forces determined once for all by external sources. By its mere moving presence, each particle becomes the source of an electromagnetic field. As a result, each particle is submitted to the combined action of the external field and of the field created by the other particles in the plasma. At any point in space, the intensity of the latter field depends on the position and momentum of the corresponding point sources: it is a violently fluctuating function of space and time. The fields created by the particles are the carriers of the *interactions* between all the particles in the plasma. The equations of motion become an enormous set of inextricably coupled non-linear equations. Even the most powerful computer is at loss when confronted with such a formidable problem.

The way out of this apparent dilemma is well-known. One must start asking different questions about the system. We shall no longer want to determine the detailed trajectory of each particle, because we are unable to observe it anyway. We shall rather ask questions about the global properties of the plasma, which are observable on a macroscopic scale (e.g., the motion of the fluid rather than of its constituent particles, the transport of energy from one point to another, the propagation of a wave through the medium, etc.) In doing so, the complexity of the system turns into an advantage. Indeed, the large number of particles and the extreme irregularity of their motion lead to the picture of a *quasi-chaotic* system, to which some of the concepts of probability theory become applicable. We thus go over from ordinary mechanics into the realm of *statistical mechanics* and of its daughter, *kinetic theory*.

It is, of course, out of the question to develop in the present book the details of statistical mechanics and of kinetic theory. Many books on this subject exist on the market (e.g., Balescu 1963, 1975, Landau and Lifshitz 1980, Lifshitz and Pitaevskii 1981, Akhiezer and Peletminskii 1981 and Klimontovich 1964, 1982). We shall assume the reader to be familiar with the elements of statistical physics; we shall, however, provide sketchy derivations of some of the key properties.

In the present chapter we study the simpler aspects of the motion of the individual charged particle in electromagnetic fields. Before really starting, we introduce the subject by two preparatory sections. Section 1.2 is a (rather unorthodox) review of Hamiltonian mechanics, where we develop a quite

important theoretical tool for the study of charged particle dynamics. In section 1.3 we recall some important concepts from differential geometry, which are useful for understanding and describing any vector field, such as the magnetic field.

1.2. *Hamiltonian mechanics. Canonical and pseudo-canonical transformations*

Hamilton's formulation of classical mechanics is a standard chapter of mathematical physics (one of the best presentations is found in the book by Goldstein 1980, see also Arnold 1978). It is, however, advantageous to look at Hamiltonian mechanics from a slightly non-standard point of view which, besides its elegance, has proved in recent years to be extremely useful in applications. We believe that the first work done in this "algebraic" spirit is the paper by Dirac (1949) on classical relativistic mechanics. His ideas were developed systematically in the review paper by Currie et al. (1963) and applied to statistical mechanics by Balescu and Kotera (1967), and by Balescu et al. (1967). Significant papers in this spirit are also those by Bialynicki-Birula (1970, 1975), Bialynicki-Birula and Iwinski (1973) and Balescu and Poulain (1974). Subsequently, Sudarshan and Mukunda (1974) published a complete textbook on classical mechanics, based on this philosophy. The applications to the present problem came from an independent direction and will be discussed in the forthcoming sections.

Any theory of a system evolving in time must be constructed upon two concepts: a definition of the *state* of the system, and a definition of the *law of evolution*. In classical Hamiltonian mechanics, the *state* of a system of f degrees of freedom requires the specification of $2f$ variables which are taken (in a first stage) as the generalized coordinates q_i and momenta p_i,

$$(q_i, p_i), \quad i = 1, 2, \ldots, f. \tag{2.1}$$

The *law of motion* is governed by a specified function of the state variables: the *Hamiltonian* $H(q, p)$. Here (q, p) is an abbreviation for the set $(q_1, \ldots, q_f, p_1, \ldots, p_f)$. Here, and in all forthcoming sections up to section 1.9, we assume the Hamiltonian to be independent of the time, t. The system is then called an *autonomous system*. The motion, i.e., the change in time of the positions and of the momenta, is determined by the Hamilton equation

$$\dot{q}_i(t) = \frac{\partial H(q, p)}{\partial p_i}, \qquad \dot{p}_i(t) = -\frac{\partial H(q, p)}{\partial q_i}. \tag{2.2}$$

Besides q_i and p_i, we are interested in many other quantities which take definite values in each state of the system. The Hamiltonian, as well as other physical quantities, such as the total momentum, the angular momentum, etc., are obvious physical examples. A subtler example is provided by the solution of eqs. (2.2). The coordinate $q_i(t)$ at time t is a function of the *initial* coordinates and momenta q_i, p_i and of the time. We decide from here on to take the set (q, p), representing the coordinates and the momenta at a given (initial) time, as the variables spanning a (fixed) *phase space* of the dynamical system. All the quantities mentioned above are then defined (in classical mechanics) as functions of the phase space coordinates (q, p), possibly depending on some "external parameters" α; they will be called *dynamical functions* and will be denoted by $a, b, \ldots$:

$$a \equiv a(q, p; \alpha). \tag{2.3}$$

Typical examples of external parameters are: the time t, parameters characterizing the system (such as the mass m, or the charge e), or characterizing the environment (such as the magnetic field $\boldsymbol{B}$), or universal constants (such as the speed of light c), or simply numerical constants.

We now consider the set $\mathscr{D}$ of all the dynamical functions and define the permissible operations on its members; in other words, we *endow the set with an algebraic structure*. We postulate that, for any member $a, b, \ldots \in \mathscr{D}$, the result of the following operations is also a member of the set $\mathscr{D}$:
Multiplication by an external parameter:

$$\alpha a = c, \quad c \in \mathscr{D}. \tag{2.4}$$

Linear combination *:

$$\alpha a + \beta b = d, \quad d \in \mathscr{D}. \tag{2.5}$$

Multiplication:

$$a \cdot b = e, \quad e \in \mathscr{D}. \tag{2.6}$$

Inversion:

$$a^{-1} = f, \quad f \in \mathscr{D}. \tag{2.7}$$

* The mathematical operation of linear combination is restricted by physical considerations. Only quantities having the same *physical dimension* can be superposed. This restriction is far from trivial, because q and p have different dimensions. Hence d exists only if there exist external parameters such that αa and βb have the same dimensions. Another restriction comes from the requirement that only quantities of the same *tensorial* nature (scalars, vectors,...) can be added.

These operations have their usual algebraic meaning and properties. Clearly, taking q and p as "building blocks", we can construct by means of these operations very wide classes of dynamical functions (e.g. polynomials, analytical functions, meromorphic functions, Fourier series, etc.). But the truly characteristic feature of the set $\mathscr{D}$ is obtained by defining an additional operation, the *Lie bracket*,

$$[a, b] = g, \quad g \in \mathscr{D}, \tag{2.8}$$

with the following properties. It is an *antisymmetric* operation:

$$[a, b] = -[b, a]. \tag{2.9}$$

It is a non-associative operation, governed by the *Jacobi relation*:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0. \tag{2.10}$$

It is related as follows to the three basic operations (2.4)–(2.6):

$$[\alpha a, b] = \alpha[a, b], \tag{2.11}$$

$$[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c], \tag{2.12}$$

$$[ab, c] = a[b, c] + b[a, c]. \tag{2.13}$$

The set $\mathscr{D}$ of dynamical functions, endowed with the operations (2.4)–(2.8), is called a *Lie algebra*.

The three rules (2.11)–(2.13) imply that the Lie bracket has properties analogous to a first order differential operator. They imply the following important, easily derived relation

$$[a(q, p), b] = \sum_{i=1}^{f} \left(\frac{\partial a}{\partial q_i}[q_i, b] + \frac{\partial a}{\partial p_i}[p_i, b] \right). \tag{2.14}$$

Iterating this relation, we find

$$\begin{aligned}[a(q, p), b(q, p)] &= \sum_{i=1}^{f} \sum_{j=1}^{f} \left(\frac{\partial a}{\partial q_i} \frac{\partial b}{\partial q_j}[q_i, q_j] + \frac{\partial a}{\partial q_i} \frac{\partial b}{\partial p_j}[q_i, p_j] \right. \\ &\quad \left. + \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial q_j}[p_i, q_j] + \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial p_j}[p_i, p_j] \right).\end{aligned} \tag{2.15}$$

This very important relation enables us to calculate explicitly the Lie bracket of any pair of functions of the phase space coordinates, provided we know the fundamental Lie brackets of the "building stones" q_i, p_i. At this stage in the Hamiltonian formalism, we *postulate* the following value of the *fundamental Lie brackets*:

$$[q_i, q_j] = 0,$$

$$[p_i, p_j] = 0,$$

$$[q_i, p_j] = \delta_{ij}, \quad i, j = 1, 2, \ldots, f. \tag{2.16}$$

For completeness, the following relations

$$[\alpha, q_i] = 0,$$

$$[\alpha, p_i] = 0, \quad i = 1, 2, \ldots, f \tag{2.17}$$

must be added for any external parameter α. Actually, eqs. (2.17) can be considered as a general *definition* of an external parameter. Equations (2.16) can be called the *Lie multiplication table* of the Lie algebra $\mathscr{D}$ in the variables (q_i, p_i). Any set of variables satisfying (2.16) is called a set of *canonical variables*. It is also said that q_i and p_i are *canonically conjugate*.

Combining (2.15) and (2.16), we are in a position of calculating explicitly the Lie bracket of any pair of dynamical functions. The former equation reduces to

$$[a(q, p), \quad b(q, p)] = \sum_{i=1}^{f} \left(\frac{\partial a}{\partial q_i} \frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial q_i} \right). \tag{2.18}$$

This is the familiar expression of the *Poisson bracket*. It appears as a specific realization of the Lie bracket, *valid when the fundamental variables q_i, p_i are canonical*.

The special importance of the Lie bracket for dynamics comes from the following postulate: *The evolution in time of any dynamical function, due to the motion of the system, is determined by the equation*

$$\dot{a} = [a, H]. \tag{2.19}$$

This is *the most general form of the equation of motion in the Hamiltonian formalism*. Using the Poisson bracket (2.18), we easily see that (2.19) reduces to Hamilton's equations (2.2) when we take for a the particular functions q_i, p_i.

We now discuss a topic that turns out to be very important for the forthcoming applications: *transformation theory*. For some theoretical or practical reasons, it may turn out that the initial choice of canonical variables (q_i, p_i) is not the most suitable one for a given problem. One may want to use some other set of variables which, for example, relates more closely to the symmetry of the problem. In all classical textbooks (e.g. Goldstein 1980) it is shown in detail that a certain class of transformations of variables plays a privileged role in the framework of Hamiltonian theory.

Consider a change of variables from (q_i, p_i) to (Q_i^c, P_i^c), defined by

$$Q_i^c = Q_i^c(q, p), \qquad P_i^c = P_i^c(q, p). \tag{2.20}$$

The transformation is called a *canonical transformation* if the new variables are again canonically conjugate:

$$\left[Q_i^c, Q_j^c\right] = 0, \qquad \left[P_i^c, P_j^c\right] = 0, \qquad \left[Q_i^c, P_j^c\right] = \delta_{ij}. \tag{2.21}$$

It is easily shown that as a consequence of these relations, the *Jacobian* J^c of a canonical transformation equals one:

$$J^c = 1. \tag{2.22}$$

This property is easily checked in the case of a single degree of freedom. We calculate, by using (2.18),

$$[Q^c, P^c] = \left(\frac{\partial Q^c}{\partial q}\frac{\partial P^c}{\partial p} - \frac{\partial Q^c}{\partial p}\frac{\partial P^c}{\partial q}\right).$$

If Q^c, P^c are to be canonical, we must have

$$[Q^c, P^c] = \begin{vmatrix} \dfrac{\partial Q^c}{\partial q} & \dfrac{\partial Q^c}{\partial p} \\ \dfrac{\partial P^c}{\partial q} & \dfrac{\partial P^c}{\partial p} \end{vmatrix} = J^c = 1.$$

A general proof is given below, see eq. (2.36).

The main property of the canonical transformations pertains to their *leaving the Hamilton equations (2.2) form-invariant*. In other words, the equations of motion for Q_i^c, P_i^c are

$$\dot{Q}_i^c = \frac{\partial H(Q^c, P^c)}{\partial P_i^c}, \qquad \dot{P}_i^c = -\frac{\partial H(Q^c, P^c)}{\partial Q_i^c}. \tag{2.23}$$

Let us also check this property for a single degree of freedom. Using (2.19) and (2.18) we have

$$\dot{Q}^c = \frac{\partial Q^c}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q^c}{\partial p}\frac{\partial H}{\partial q}.$$

We now note

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q^c}\frac{\partial Q^c}{\partial p} + \frac{\partial H}{\partial P^c}\frac{\partial P^c}{\partial p}$$

with a similar formula for $\partial H/\partial q$. Substituting these relations in the right-hand side of the previous equation, we obtain

$$\dot{Q}^c = \frac{\partial H}{\partial P^c}\left(\frac{\partial Q^c}{\partial q}\frac{\partial P^c}{\partial p} - \frac{\partial Q^c}{\partial p}\frac{\partial P^c}{\partial q}\right) = \frac{\partial H}{\partial P^c}[Q^c, P^c] = \frac{\partial H}{\partial P^c},$$

the last equality following from (2.21).

The literature on Hamiltonian dynamics is dominated by canonical transformations. This is, however, a drawback, because it is often impossible to use physically convenient variables which are at the same time canonically conjugate. Only in recent years was it realized explicitly that the really important feature of Hamiltonian mechanics is the algebraic structure of the set $\mathscr{D}$ of dynamical functions, together with the dynamical law (2.19), rather than the particular form (2.2). Indeed, the operations (2.4)–(2.8) and the dynamical law (2.19) are formulated in a form that is *independent of any particular choice of phase space variables* (or "building stones"). They can be translated consistently into any set of coordinates, whether canonical or not. The importance of the freedom and flexibility provided by the use of non-canonical variables is particularly striking in electromagnetic problems. This was the main motivation of the first works using non-canonical Hamiltonian mechanics (Bialynicki-Birula 1970, 1975, Bialynicki-Birula and Iwinski 1973, Balescu and Poulain 1974 and Littlejohn 1979, 1981, 1983). Let us see how the method is made operational.

The starting point is the definition of a dynamical system as a Lie algebra $\mathscr{D}$ and its initial realization in terms of a set of canonical coordinates (q_i, p_i). [In other words, everything we said up to eq. (2.19) is valid as it stands]. We now introduce an arbitrary, invertible change of variables to (Q_i, P_i):

$$Q_i = Q_i(q, p), \qquad P_i = P_i(q, p). \tag{2.24}$$

The new variables (Q_i, P_i) *are, in general, not canonical.* In other words, eqs. (2.21) do not hold for the new variables. On the other hand, the definitions (2.24) and the rules (2.9)–(2.15) allow us to calculate explicitly the *fundamental Lie brackets* of the new variables:

$$\begin{aligned}
&[Q_i, Q_j] = \mathscr{F}_{ij}(Q, P),\\
&[P_i, P_j] = \mathscr{G}_{ij}(Q, P),\\
&[Q_i, P_j] = \mathscr{H}_{ij}(Q, P), \quad i, j = 1, 2, \ldots, f.
\end{aligned} \tag{2.25}$$

Of course, we still have

$$\begin{aligned}
&[\alpha, Q_i] = 0,\\
&[\alpha, P_i] = 0, \quad i = 1, 2, \ldots, f.
\end{aligned} \tag{2.26}$$

We note the necessary relations

$$\mathscr{F}_{ij} = -\mathscr{F}_{ji}, \qquad \mathscr{G}_{ij} = -\mathscr{G}_{ji}, \qquad [P_i, Q_j] = -\mathscr{H}_{ji}, \tag{2.27}$$

which result from the antisymmetry of the Lie bracket. Equations (2.25) constitute the Lie multiplication table of $\mathscr{D}$ in the variables (Q_i, P_i). The main difference with (2.16) lies in the fact that *the fundamental Lie brackets are no longer constants* (0 *or* 1), *but are non-trivial dynamical functions.*

For the simplicity of language, we propose to call any transformation of phase space variables $(q_i, p_i) \to (Q_i, P_i)$ a *pseudo-canonical transformation* whenever it is specified as follows:

(a) (Q_i, P_i) are related to (q_i, p_i) by a set of $2f$ invertible (sufficiently regular) point functions (2.24);

(b) a Lie multiplication table of the set (Q_i, P_i) is specified by (2.25), consistently with the Lie multiplication table of the initial variables (q_i, p_i). In other words, the Lie algebra structure of (q_i, p_i) is "imprinted" upon (Q_i, P_i).

This implies, specifically, the following operations. For calculating $[Q_i, P_j]$, say, one first considers Q_i and P_j as functions of the old variables (q, p); one calculates the Lie bracket of these dynamical functions by the rules of the Lie algebra in the (q, p) variables; the resulting dynamical function $h_{ij}(q, p)$ is expressed in terms of the new variables by using the inverse transformation of (2.24),

$$h_{ij}(q(Q, P), p(Q, P)) = \mathscr{H}_{ij}(Q, P).$$

A pseudo-canonical transformation reduces to a canonical one whenever the Lie multiplication table of the new variables is of the form (2.21).

We have seen above that the *Jacobian* of a canonical transformation equals one (see 2.22); this property is lost for a pseudo-canonical transformation (2.24). We now show that the Jacobian of the latter transformation is simply related to the fundamental Lie brackets (Littlejohn 1979). We introduce the $2f$-dimensional vectors

$$z^i = (q_1, \ldots, q_f, p_1, \ldots, p_f),$$

$$Z^i = (Q_1, \ldots, Q_f, P_1, \ldots, P_f), \quad i = 1, 2, \ldots, 2f. \tag{2.28}$$

We also consider the matrices σ and Σ, whose elements are

$$\sigma^{ij} = [z^i, z^j],$$

$$\Sigma^{ij} = [Z^i, Z^j], \quad i, j = 1, 2, \ldots, 2f. \tag{2.29}$$

They are simply the matrices associated with the fundamental Lie brackets. Clearly, the variables (q, p) being canonical, we have

$$\sigma = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix}, \tag{2.30}$$

where $\boldsymbol{I}$ is the $f \times f$ unit matrix. We also have

$$\Sigma = \begin{pmatrix} \mathscr{F} & \mathscr{H} \\ -\mathscr{H}^T & \mathscr{G} \end{pmatrix}, \tag{2.31}$$

where $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ are the $f \times f$ matrices whose elements are defined by (2.25), and $\mathscr{H}^{\mathrm{T}}$ is the transpose of $\mathscr{H}$. The relation between the matrices Σ and σ is found by considering $Z = Z(z)$, defined by (2.24) and using the Poisson bracket (2.18),

$$\Sigma^{ij} = [Z^i, Z^j] = \sum_{k=1}^{f} \left(\frac{\partial Z^i}{\partial q_k} \frac{\partial Z^j}{\partial p_k} - \frac{\partial Z^i}{\partial q_k} \frac{\partial Z^j}{\partial p_k} \right) = \sum_{m=1}^{2f} \sum_{n=1}^{2f} \frac{\partial Z^i}{\partial z^m} \sigma^{mn} \frac{\partial Z^j}{\partial z^n}. \tag{2.32}$$

This relation demonstrates the remarkable property that the matrix of the fundamental Lie brackets transforms under (2.24) as a contravariant second-rank tensor. Calculating now the determinant of both sides, we have

$$\| \Sigma \| = \| \sigma \| \left(\left\| \frac{\partial Z^k}{\partial z^m} \right\| \right)^2, \tag{2.33}$$

but, clearly,

$$\| \sigma \| = 1. \tag{2.34}$$

On the other hand, the determinant between parentheses in (2.33) is simply the inverse of the Jacobian J of the transformation $z \to Z$ [or $(q, p) \to (Q, P)$]:

$$\left\| \frac{\partial Z^k}{\partial z^m} \right\| = \frac{1}{J}. \tag{2.35}$$

We thus obtain from (2.33)

$$J^2 = \frac{1}{\| \Sigma \|}. \tag{2.36}$$

This remarkable relation allows us to calculate very simply the Jacobian of the pseudo-canonical transformation from the Lie multiplication table. Clearly, (2.36) reduces to (2.22) for a canonical transformation, because of (2.34).

Once the fundamental Lie brackets are determined, the Lie bracket of any pair of dynamical functions can be calculated explicitly by using the rules (2.9)–(2.13). Note, in particular, that (2.15) is still valid with the variables (Q, P) replacing (q, p); indeed it is a direct consequence of the rules (2.11)–(2.13), which do not require the variables (q, p) to be canonical. Using (2.25), eq. (2.15) is easily rearranged into

$$\begin{aligned} &[a(Q, P), b(Q, P)] \\ &= \sum_{i=1}^{f} \sum_{j=1}^{f} \left[\frac{\partial a}{\partial Q_i} \frac{\partial b}{\partial Q_j} \mathscr{F}_{ij}(Q, P) + \left(\frac{\partial a}{\partial Q_i} \frac{\partial b}{\partial P_j} - \frac{\partial a}{\partial P_j} \frac{\partial b}{\partial Q_i} \right) \mathscr{H}_{ij}(Q, P) \right. \\ &\qquad \left. + \frac{\partial a}{\partial P_i} \frac{\partial b}{\partial P_j} \mathscr{G}_{ij}(Q, P) \right]. \end{aligned} \tag{2.37}$$

This formula provides us with an *explicit realization of the Lie bracket in arbitrary, non-canonical variables*. Note that in the particular case when (Q, P) are canonical [i.e. $\mathscr{F}_{ij} = \mathscr{G}_{ij} = 0$, $\mathscr{K}_{ij} = \delta_{ij}$], eq. (2.37) reduces to the Poisson bracket (2.18).

Finally, the equations of motion are given directly by (2.19):

$$\dot{Q}_i = [Q_i, H(Q, P)], \qquad \dot{P}_i = [P_i, H(Q, P)]. \tag{2.38}$$

Using (2.37) these equations can also be written in the explicit, though less compact form

$$\dot{Q}_i = \sum_{j=1}^{f} \left(\mathscr{F}_{ij}(Q, P) \frac{\partial H(Q, P)}{\partial Q_j} + \mathscr{K}_{ij}(Q, P) \frac{\partial H(Q, P)}{\partial P_j} \right),$$

$$\dot{P}_i = \sum_{j=1}^{f} \left(-\mathscr{K}_{ij}(Q, P) \frac{\partial H(Q, P)}{\partial Q_j} + \mathscr{G}_{ij}(Q, P) \frac{\partial H(Q, P)}{\partial P_j} \right). \tag{2.39}$$

These are the generalizations of the Hamilton equations for a set of non-canonical variables. They reduce to eqs. (2.21) or (2.2) when (Q, P) are canonical.

An elegant, but less explicit formula is obtained by using notations (2.28)–(2.31),

$$\dot{Z}^j = \Sigma^{jk}(Z) \frac{\partial}{\partial Z^k} H(Z). \tag{2.40}$$

1.3. *Magnetic field and magnetic field lines. Intrinsic local reference frame*

The dynamical systems of major interest to us consist of charged particles moving in the presence of an electric and a magnetic field. The former is a rather "classical" force field, not very different from gravitation; it does not require a special discussion at this point. Magnetic fields, on the other hand, are rather "unorthodox": they act on the particles through the Lorentz force, which depends on the velocity and is directed perpendicularly to the field. Given the importance of magnetic fields in plasma physics, we give here a first geometrical discussion of the magnetic field, which prepares the dynamical studies to follow. This discussion is further amplified in chapter 8.

We consider a region of space, bounded or infinite, which is permeated by a magnetic field. We do not bother at this point about the sources of this field.

In order to do analytical geometry, we need to define a reference frame, whic allows us to characterize each point in space by a set of three numbers, its *coordinates*. This reference frame is defined by a triad of mutually perpendicular unit vectors (i.e. a *basis*) located at a point O called the origin. The position of the origin and the orientation of the basis vectors can be chosen quite arbitrarily. We call this the *absolute reference frame*. In this frame, each point is defined by its three Cartesian coordinates (x_1, x_2, x_3), or, equivalently, by the vector $\boldsymbol{x}$ going from the origin O to the point considered.

The magnetic field is a *vector field*: with each point in space, it associates a vector $\boldsymbol{B}(\boldsymbol{x})$. We assume here that the field is *static*, i.e. time-independent. The components of the magnetic field along the directions of the absolute reference frame will be denoted by $B_i(\boldsymbol{x})$ $(i = 1, 2, 3)$. The $B_i(\boldsymbol{x})$ are assumed to be continuous and twice differentiable, with the possible exception of a finite number of singular points. Alternatively, one may say that, in each point, the magnetic field is characterized by a scalar (a non-negative number) called the *intensity* (or strength), defined as the length of the vector:

$$B(\boldsymbol{x}) = |\boldsymbol{B}(\boldsymbol{x})| = \left(\sum_{i=1}^{3} B_i^2 \right)^{1/2}, \tag{3.1}$$

and by a *unit vector* $\boldsymbol{b}(\boldsymbol{x})$, defining a characteristic direction in each point,

$$\boldsymbol{b}(\boldsymbol{x}) = \frac{1}{B(\boldsymbol{x})} \boldsymbol{B}(\boldsymbol{x}). \tag{3.2}$$

We may thus write alternatively

$$\boldsymbol{B}(\boldsymbol{x}) = B(\boldsymbol{x})\ \boldsymbol{b}(\boldsymbol{x}). \tag{3.3}$$

In general, both the intensity and the direction of the unit vector vary in space. We may, however, consider the following particular cases, which occur when one or the other factor is constant:

$$\boldsymbol{B}(\boldsymbol{x}) = B\boldsymbol{b} \qquad \text{homogeneous field} \tag{3.4}$$

$$\boldsymbol{B}(\boldsymbol{x}) = B(\boldsymbol{x})\boldsymbol{b} \qquad \text{straight inhomogeneous field} \tag{3.5}$$

$$\boldsymbol{B}(\boldsymbol{x}) = B\boldsymbol{b}(\boldsymbol{x}) \qquad \text{curved field of constant intensity} \tag{3.6}$$

(the concepts of straightness and of curvature will be defined presently).

We now note that a vector field can also be described in a different (though equivalent) way. We start at a given point $\boldsymbol{x}$, at which the field has the value

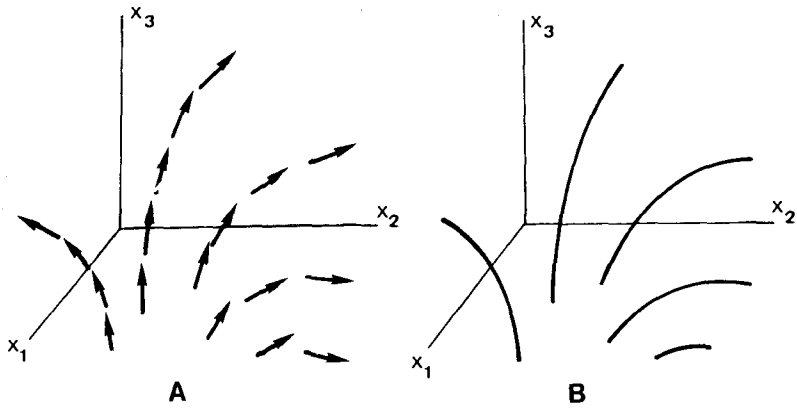

Fig. 3.1. The two points of view in describing a vector field. (A) Set of vectors. (B) Set of curves (field lines).

$\boldsymbol{B}(x)$. We define a small (infinitesimal) section of a curve, which is tangent to the vector $\boldsymbol{B}(x)$ at x. At the end, $x + \mathrm{d}x$, of this small section we add another section which is tangent to the vector $\boldsymbol{B}(x + \mathrm{d}x)$, and continue this process indefinitely. We have thus associated with the initial point x an infinite (one-parameter) set of points, i.e. a *curve*, which is tangent at each point to the local vector of the field. We thus arrive at a view of the space in which the points are no longer independent, but are structured as a set of curves, called *field lines* (fig. 3.1). Alternatively, one may say that through each point passes a field line, defined parametrically in the form $x = x(\tau)$, where τ is an arbitrary parameter. In order to express the tangency condition, these curves must satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \boldsymbol{B}(x(\tau)) \equiv \tilde{\boldsymbol{B}}(\tau), \tag{3.7}$$

which can also be expressed in the more familiar form

$$\frac{\mathrm{d}x_1}{\tilde{B}_1} = \frac{\mathrm{d}x_2}{\tilde{B}_2} = \frac{\mathrm{d}x_3}{\tilde{B}_3} = \mathrm{d}\tau. \tag{3.8}$$

The particular field line passing through the given point x is the solution of the differential equation (3.7), satisfying the initial condition

$$x(\tau = 0) = x. \tag{3.9}$$

We now introduce some important concepts from the differential geometry of curves (see e.g. Smirnov 1970, Vol. 2, Doubrovine et al. 1982, Vol. 1). The length s_{AB} of an arc of the curve (3.7) is defined as

$$s_{AB} = \int_{\tau_A}^{\tau_B} d\tau \left| \frac{dx}{d\tau} \right| = \int_{\tau_A}^{\tau_B} d\tau \, \tilde{B}(\tau) \equiv \int_{s_A}^{s_B} ds, \tag{3.10}$$

where we introduced the infinitesimal arc length ds by

$$ds = \tilde{B}(\tau) \, d\tau. \tag{3.11}$$

The parameter s is the *natural parameter*. When the equation of the curve is expressed as a function of s, all the important concepts take a particularly simple form. Thus, from (3.7) and (3.11) we obtain

$$\frac{dx(s)}{ds} = \frac{d\tau}{ds} \frac{dx}{d\tau} = \frac{1}{\tilde{B}} \tilde{\boldsymbol{B}}.$$

Hence, the equation of the field line, expressed in terms of the natural parameter, is

$$\frac{dx(s)}{ds} = \boldsymbol{b}(x(s)) \equiv \hat{\boldsymbol{b}}(s). \tag{3.12}$$

This result also shows that (dx/ds) is precisely the unit vector along the tangent to the curve at s. In the case when $\boldsymbol{b}$ reduces to a constant, the tangent points in a constant direction, and the field line is a straight line: one says that the field is *straight* (see eqs. 3.4 and 3.5).

Consider now another field $F(x)$ defined in space. This field may be of whatever nature: scalar, vector, etc. We now ask how this field varies when x travels along a magnetic field line. In this case, $F(x)$ becomes a function $\hat{F}(s)$ of the parameter s,

$$F(x) \Rightarrow F(x(s)) \equiv \hat{F}(s).$$

It is often important to calculate the derivative of $\hat{F}(s)$ with respect to s, i.e., the derivative of $F(x)$ along the field line. Clearly,

$$\frac{d\hat{F}(s)}{ds} = \frac{dx}{ds} \cdot \frac{\partial F(x)}{\partial x} = [\boldsymbol{b}(x) \cdot \nabla] F(x). \tag{3.13}$$

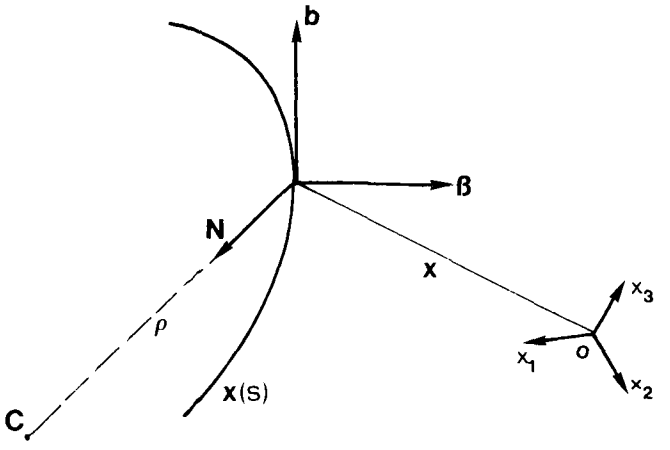

Fig. 3.2. The Frenet triad at point x. It consists of the unit vectors N along the principal normal, β along the binormal and b along the tangent. The centre of curvature C is at a distance ρ from x. Also shown is the absolute reference frame (x_1, x_2, x_3).

We now come back to (3.12). As a constant $\boldsymbol{b}$ corresponds to a straight line, it is intuitively clear that the rate of change of $\boldsymbol{b}$ is related to the concept of *curvature*. We therefore define the *curvature vector* $\boldsymbol{k}$ as

$$\hat{\boldsymbol{k}}(s) \equiv \frac{\mathrm{d}\,\hat{\boldsymbol{b}}(s)}{\mathrm{d}s} = \frac{\mathrm{d}^2 \boldsymbol{x}(s)}{\mathrm{d}s^2}. \tag{3.14}$$

This vector is perpendicular to $\boldsymbol{b}$ and points toward the concave side of the curve (fig. 3.2). This follows from the fact that $\hat{\boldsymbol{b}}(s)$ is a unit vector, hence

$$\hat{\boldsymbol{b}} \cdot \frac{\mathrm{d}}{\mathrm{d}s}\hat{\boldsymbol{b}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\hat{b}^2(s) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}1 = 0.$$

The length of this vector $\hat{\boldsymbol{k}}(s)$ defines the *radius of curvature* ρ,

$$\frac{1}{\rho} = \left|\frac{\mathrm{d}\hat{\boldsymbol{b}}(s)}{\mathrm{d}s}\right|. \tag{3.15}$$

Note that, by definition, ρ is a non-negative number. The direction of the

curvature vector defines the *principal normal*, characterized by the unit vector $\hat{N}(s)$,

$$\hat{k}(s) = \frac{1}{\hat{\rho}(s)}\hat{N}(s). \tag{3.16}$$

We also define a point $C(x)$ lying on the principal normal at a distance $\rho(x)$ from x on the concave side. This point is called the *centre of curvature*. An infinitesimal arc of the curve in the neighbourhood of x can be assimilated to an infinitesimal arc of a circle of radius $\rho(x)$, centered at $C(x)$.

We now switch to the "field description" by using (3.13), applied to $F(x) = b(x)$. In this way, we associate with $b(x)$ a new vector field $N(x)$ through

$$\frac{\mathrm{d}\hat{b}(s)}{\mathrm{d}s} = [b(x)\cdot\nabla]\,b(x) = \frac{1}{\rho(x)}N(x). \tag{3.17}$$

The local radius of curvature is

$$\frac{1}{\rho(x)} = |(b\cdot\nabla)b|. \tag{3.18}$$

Having defined in each point two mutually perpendicular unit vectors b and N, it is easy to define a third one, β, perpendicular to both. It is called the unit vector along the *binormal*,

$$\beta(x) = b(x)\wedge N(x). \tag{3.19}$$

If this vector is constant, it is easily seen that the curve $x(s)$ lies in a plane. Therefore, the derivative of $\hat{\beta}(s) \equiv \beta[x(s)]$ measures the rate at which the field line "goes out of the plane" and is therefore called the *torsion vector* κ. It is easily proved from (3.19) that this vector is directed along the principal normal, thus

$$\kappa = \frac{\mathrm{d}\hat{\beta}(s)}{\mathrm{d}s} = [b(x)\cdot\nabla]\,\beta(x) = \frac{1}{\tau(x)}N(x). \tag{3.20}$$

The *radius of torsion* τ is thus defined as

$$\frac{1}{\tau} = N\cdot[(b\cdot\nabla)\beta]. \tag{3.21}$$

Unlike ρ, the radius of torsion τ may be negative.

The derivative of $\boldsymbol{N}$ with respect to s does not introduce any new quantity, but it is related to the ones already defined, as can easily be shown. We collect here the three relations which are celebrated under the name of *Frenet's formulae*:

$$\frac{\mathrm{d}\hat{\boldsymbol{b}}(s)}{\mathrm{d}s} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = \frac{1}{\rho}\boldsymbol{N}, \tag{3.22}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{N}}(s)}{\mathrm{d}s} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{N} = -\frac{1}{\tau}\boldsymbol{\beta} - \frac{1}{\rho}\boldsymbol{b}, \tag{3.23}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\beta}}(s)}{\mathrm{d}s} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{\beta} = \frac{1}{\tau}\boldsymbol{N}, \tag{3.24}$$

The main result of this discussion is the following. Given an inhomogeneous vector field $\boldsymbol{B}(\boldsymbol{x})$, we have been able to attach to each point not only one, but *three* characteristic directions, the orientation of which is entirely determined by the geometry of the field. In other words, there exists at each point an orthogonal right-handed triad, called the *Frenet triad* $[\boldsymbol{N}(\boldsymbol{x}), \boldsymbol{\beta}(\boldsymbol{x}), \boldsymbol{b}(\boldsymbol{x})]$ with the relations

$$\boldsymbol{N} = \boldsymbol{\beta} \wedge \boldsymbol{b}, \qquad \boldsymbol{\beta} = \boldsymbol{b} \wedge \boldsymbol{N}, \qquad \boldsymbol{b} = \boldsymbol{N} \wedge \boldsymbol{\beta}. \tag{3.25}$$

This is a very important result, because it allows us to define in each point $\boldsymbol{x}$ a *local reference frame* which is perfectly adapted to the geometry of the magnetic field and provides us with the natural frame for the description of the phenomena in a magnetic field.

We note, however, that this situation degenerates in the case of a *straight field*, i.e., when the radius of curvature becomes infinite,

$$(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = 0. \tag{3.26}$$

In this case, we see from (3.17) that the direction of the principal normal (hence also of the binormal) is no longer determined. In other words, whereas one direction, $\boldsymbol{b}$, is still fixed at each point, the two other directions, $\boldsymbol{N}$ and $\boldsymbol{\beta}$, become arbitrary. In this case we are still able to define a local reference frame by the following conventions.

(a) Straight inhomogeneous field (3.5). In this case, there exists, besides $\boldsymbol{b}$, another intrinsic local direction fixed by the field: the gradient of its intensity.

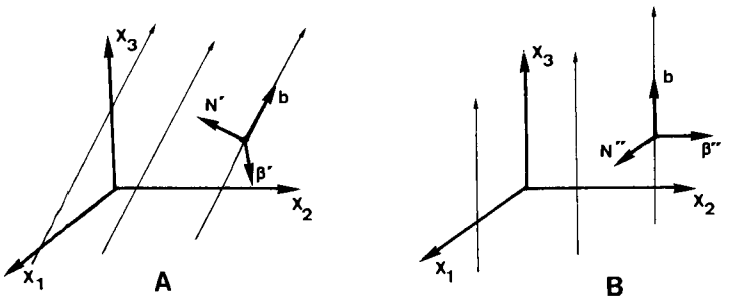

Fig. 3.3. Degenerate local triads. (A) Straight inhomogeneous field: the local vector N' points along $b \wedge \nabla B(x)$. (B) Homogeneous field: the local triad is everywhere parallel to the absolute one.

We introduce the unit vector N' defined as

$$N'(x) = \frac{b(x) \wedge \nabla B(x)}{|\nabla B(x)|} \tag{3.27}$$

and a third vector

$$\beta'(x) = b(x) \wedge N'(x). \tag{3.28}$$

We can thus again construct an intrinsic local triad entirely determined by the geometry of the field (see fig. 3.3). In this case the local triad has one vector, b, pointing in a constant direction, and two others which, in general, turn around the axis b as the point x moves along a field line.

Note, however, that this construction is possible only when the gradient of $B(x)$ is not everywhere parallel to the direction of the field. In the particular case when

$$b \wedge \nabla B(x) = 0 \quad \forall x \tag{3.29}$$

the gradient of B no longer defines a direction in space distinct of b, and we are in a situation similar to the homogeneous field.

(b) Homogeneous field (3.4), or *straight field* (3.5) *with* $b \wedge \nabla B = 0$. In this case, there is nothing in the field which allows us to define more than one characteristic direction, the same in all space. We have absolute cylindrical symmetry: all directions perpendicular to b are equivalent. We may therefore choose the absolute reference frame in such a way that the direction Ox_3 is parallel to the constant field, and choose the direction Ox_1 (hence Ox_2) arbitrarily. We then *define* the local triad as being everywhere parallel to the absolute frame.

1.4. *Equations of motion of a charged particle in an inhomogeneous stationary electromagnetic field. Particle variables*

We now begin the study of the motion of a *single point particle* of mass m and charge e. The particle moves under the action of a *given* (external) magnetic field $\boldsymbol{B}(\boldsymbol{x})$ and of a *given* (external) electric field $\boldsymbol{E}(\boldsymbol{x})$. It is assumed here that both fields may be *inhomogeneous* (i.e. may depend on the position $\boldsymbol{x}$ in space), but that they are *stationary*, i.e. their value at any position does not depend on time. These static fields can also be characterized by a vector potential $\boldsymbol{A}(\boldsymbol{x})$ and a scalar potential $\Phi(\boldsymbol{x})$, to which they are related by the well-known relations

$$\boldsymbol{B}(\boldsymbol{x}) = \nabla \wedge \boldsymbol{A}(\boldsymbol{x}), \tag{4.1}$$

$$\boldsymbol{E}(\boldsymbol{x}) = -\nabla \Phi(\boldsymbol{x}). \tag{4.2}$$

The derivation of the equations of motion from Hamilton's variational principle is a classical matter, treated in most textbooks on electromagnetic theory or mechanics (e.g. Landau and Lifshitz, 1957, Goldstein, 1980). We do not reproduce it here, but briefly recall the results.

Our dynamical system has three degrees of freedom ($f = 3$). We define the three coordinates q_1, q_2, q_3 as the Cartesian coordinates of the spatial point where the particle is located (briefly: the *position of the particle*). These three coordinates q_i ($i = 1, 2, 3$) are the components of a vector $\boldsymbol{q}$. The remaining three phase space variables, i.e. the components p_i ($i = 1, 2, 3$) of the *momentum $\boldsymbol{p}$ of the particle* must be defined via the Lagrangian [$p_i = \partial L / \partial \dot{q}_i$] in order to ensure the correct canonical conjugation of q_i and p_i. The Hamiltonian of the system is well-known:

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2m} \left| \boldsymbol{p} - \frac{e}{c} \boldsymbol{A}(\boldsymbol{q}) \right|^2 + e\Phi(\boldsymbol{q}). \tag{4.3}$$

From this Hamiltonian, expressed in canonical variables, the equations of motion are immediately obtained from (2.2). For the sake of easy reference, we collect all the relevant equations in table 4.1.

One immediately sees that the canonical variables are indeed very inconvenient in this problem. The main reason is that the canonical momentum $\boldsymbol{p}$ does not have any simple physical meaning. In particular, it cannot be identified with the *mechanical momentum* $\boldsymbol{\pi}$, defined as usual by $\boldsymbol{\pi} = m\boldsymbol{v}$, where $\boldsymbol{v}$ is the *velocity* of the particle. From eq. (3) in table (4.1) we have

$$\boldsymbol{\pi} = m\boldsymbol{v} = \boldsymbol{p} - \frac{e}{c} \boldsymbol{A}(\boldsymbol{q}). \tag{4.4}$$

For the same reason, we do not see the Lorentz force appear clearly in eq. (4).

Table 4.1
Canonical particle variables $\boldsymbol{q}$, $\boldsymbol{p}$.

Fundamental Lie brackets

$$[q_i, q_j] = 0, \qquad [q_i, p_j] = \delta_{ij}, \qquad [p_i, p_j] = 0. \tag{1}$$

Hamiltonian

$$H = \frac{1}{2m}\left|\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right|^2 + e\Phi. \tag{2}$$

Equations of motion

$$\dot{q}_i = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right), \tag{3}$$

$$\dot{p}_i = \frac{e}{mc}\left(p_j - \frac{e}{c}A_j\right)\nabla_i A_j - e\nabla_i \Phi. \tag{4}$$

Remarks
All fields $\boldsymbol{A}$, Φ are evaluated at $\boldsymbol{x} = \boldsymbol{q}$; $\nabla_i \equiv (\partial/\partial q_i)$.

It is therefore tempting to use the mechanical momentum $\boldsymbol{\pi}$, or the velocity $\boldsymbol{v}$ as a suitable phase space variable, instead of $\boldsymbol{p}$ (leaving the position $\boldsymbol{q}$ unchanged). But it is easily seen that the transformation $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{q}, \boldsymbol{v})$ is *not* canonical. Indeed, let us calculate, for instance,

$$[v_1, v_2] = \sum_{k=1}^{3}\left(\frac{\partial v_1}{\partial q_k}\frac{\partial v_2}{\partial p_k} - \frac{\partial v_1}{\partial p_k}\frac{\partial v_2}{\partial q_k}\right)$$

$$= \frac{e}{m^2 c}\sum_k\left[-(\nabla_k A_1)\delta_{k2} + \delta_{k1}(\nabla_k A_2)\right]$$

$$= \frac{e}{m^2 c}(\nabla_1 A_2 - \nabla_2 A_1) = \frac{e}{m^2 c}B_3(\boldsymbol{q}),$$

where we made use of (4.1). We therefore need, even in this simple case, to use the *Hamiltonian formalism in non-canonical variables* developed in section 1.2, taking $Q_i = q_i$, $P_i = v_i$. We follow the steps indicated in section 1.2 and calculate successively the fundamental Lie brackets, the Jacobian of the transformation $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{q}, \boldsymbol{v})$, the Hamiltonian expressed in terms of the new variables, and finally the equations of motion, evaluating the Lie brackets in (2.38) by means of (2.37). The results are collected in table 4.2.

Table 4.2
Particle variables $\boldsymbol{q}$, $\boldsymbol{v}$.

Definition

$$\boldsymbol{q} = \boldsymbol{q}, \quad \boldsymbol{v} = \frac{1}{m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right). \tag{1}$$

Fundamental Lie brackets

$$[q_i, q_j] = 0, \qquad [q_i, v_j] = \frac{1}{m}\delta_{ij}, \qquad [v_i, v_j] = \frac{e}{m^2 c}\varepsilon_{ijk}B_k. \tag{2}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{q}, \boldsymbol{v})$

$$J = m^3. \tag{3}$$

Hamiltonian

$$H = \tfrac{1}{2}mv^2 + e\Phi. \tag{4}$$

Equations of motion

$$\dot{\boldsymbol{q}} = \boldsymbol{v}, \tag{5}$$

$$\dot{\boldsymbol{v}} = \frac{e}{mc}(\boldsymbol{v} \wedge \boldsymbol{B}) + \frac{e}{m}\boldsymbol{E}. \tag{6}$$

Remarks
All fields: $\boldsymbol{A}$, Φ, $\boldsymbol{B}$, $\boldsymbol{E}$, are evaluated at $\boldsymbol{x} = \boldsymbol{q}$.

Now, the equations of motion are perfectly transparent. We clearly recognize in eq. (6) of table 4.2 the *Lorentz force*

$$\boldsymbol{F} = e(\boldsymbol{E} + c^{-1}\boldsymbol{v} \wedge \boldsymbol{B}). \tag{4.5}$$

The Hamiltonian (4) also has a clear form, being the sum of the kinetic and the potential energy. Here we may discuss an interesting feature. The Hamiltonian (4) is independent of the magnetic field. This is physically clear: as the magnetic term in the Lorentz force is always perpendicular to the velocity of the particle, the magnetic field does no work. How then does the magnetic field enter the equations of motion (6)? The answer is: through the *non-canonical Lie brackets* (2). Indeed, we may directly calculate

$$\dot{v}_i = \left[v_i, \tfrac{1}{2}mv^2 + e\Phi\right] = mv_j[v_i, v_j] + e\,\nabla_j\Phi[v_i, q_j]$$

$$= \frac{e}{mc}v_j\varepsilon_{ijk}B_k - \frac{e}{m}\nabla_i\Phi = \frac{e}{mc}(\boldsymbol{v} \wedge \boldsymbol{B})_i + \frac{e}{m}E_i.$$

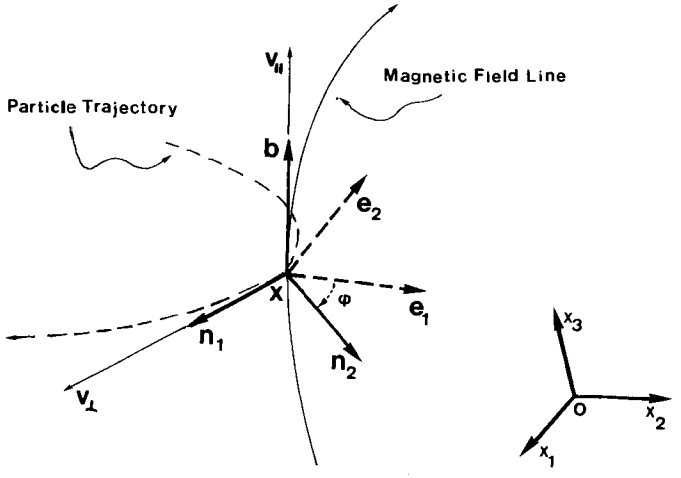

Fig. 4.1. Absolute reference frame (x_1, x_2, x_3), fixed local reference frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$, and moving local reference frame $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{b})$.

We now proceed to a third choice of variables, which is useful for the following reason. It is clear from eq. (6) that the components of the velocity parallel ($v_{\parallel}$) and perpendicular ($v_{\perp}$) to the magnetic field will play very different roles, in as much as the Lorentz force only depends on the latter. Writing $\boldsymbol{v} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}$, we wish to derive equations of motion for the two separate components. In so doing, we necessarily arrive at a *local* description, because in a non-uniform field, the orientation of these two components varies from point to point. We must carefully define the geometry of the problem (fig. 4.1), using the results of section 1.3. We consider first an *"absolute" reference frame* (x_1, x_2, x_3) with an arbitrary origin O, which is fixed once for all in space. Next, we define a *local reference frame* attached to each point of (absolute) coordinates $\boldsymbol{x}$; the orientation of this frame varies, in general, from one point to another. Its definition involves the specification of a right-handed triad of mutually orthogonal unit vectors at each point. We want this local reference frame to be *intrinsically* determined by the geometry of the magnetic field line passing through that point. The discussion in the previous paragraph provides us with the answer. In all cases, one of the vectors of the triad is the tangent unit vector $\boldsymbol{b}(\boldsymbol{x})$ along the field. Next, we must choose two additional unit vectors $\boldsymbol{e}_1(\boldsymbol{x})$, $\boldsymbol{e}_2(\boldsymbol{x})$, both perpendicular to $\boldsymbol{b}(\boldsymbol{x})$, which complete a right-handed triad

$$\boldsymbol{e}_1(\boldsymbol{x}) \wedge \boldsymbol{e}_2(\boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{x}). \tag{4.6}$$

On the basis of the results of section 1.3, we define these vectors as follows.

(a) General inhomogeneous and curved field:

$$\boldsymbol{e}_1(\boldsymbol{x}) = \boldsymbol{N}(\boldsymbol{x}), \qquad \boldsymbol{e}_2(\boldsymbol{x}) = \boldsymbol{\beta}(\boldsymbol{x}), \tag{4.7}$$

where $\boldsymbol{N}$ and $\boldsymbol{\beta}$ are unit vectors directed along the principal normal and the binormal of the magnetic field line, respectively.

(b) Straight inhomogeneous field:

$$\boldsymbol{e}_1(\boldsymbol{x}) = \frac{\boldsymbol{b} \wedge \nabla B(\boldsymbol{x})}{|\nabla B(\boldsymbol{x})|}, \qquad \boldsymbol{e}_2(\boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{x}) \wedge \boldsymbol{e}_1(\boldsymbol{x}). \tag{4.8}$$

(c) Homogeneous field, or straight field with $\boldsymbol{b} \wedge \nabla B = 0$:

The absolute reference frame is rotated in such a way as to have its Ox_3 axis parallel to the (constant) direction of the magnetic field $\boldsymbol{b}$. We then choose

$$\boldsymbol{e}_1(\boldsymbol{x}) = \boldsymbol{i}, \qquad \boldsymbol{e}_2(\boldsymbol{x}) = \boldsymbol{j}, \tag{4.9}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the constant unit vectors along Ox_1 and Ox_2 in the absolute frame. The arbitrariness of the orientation of these two vectors will not affect the observable results.

The local reference frame is fixed once for all by the geometry of the external field and is independent of the presence or absence of a particle in space. We now introduce a *third reference frame* which is also local, but whose orientation depends on the dynamics of the particle. Suppose the particle passes at some time through the point $\boldsymbol{x} = \boldsymbol{q}$, with a velocity $\boldsymbol{v}$. The latter is decomposed into components parallel and perpendicular to $\boldsymbol{b}$: $\boldsymbol{v} = v_{\parallel}\boldsymbol{b} + \boldsymbol{v}_{\perp}$. We now choose a unit vector $\boldsymbol{n}_1$ directed along $\boldsymbol{v}_{\perp}$, and a second unit vector $\boldsymbol{n}_2$ perpendicular to both $\boldsymbol{v}_{\perp}$ and to $\boldsymbol{b}$, such that

$$\boldsymbol{n}_1 \wedge \boldsymbol{n}_2 = \boldsymbol{b}. \tag{4.10}$$

We have thus defined a right-handed triad which (contrary to $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{b}$) is determined by both the external field geometry and by the particle velocity: we therefore call it the *moving local reference frame* (for want of a better name). Let us call φ the angle between the fixed unit vector $\boldsymbol{e}_1$ and the moving unit vector $\boldsymbol{n}_2$ (not $\boldsymbol{n}_1$!). A little trigonometry leads to the following relation between moving and fixed unit vectors:

$$\begin{aligned} \boldsymbol{n}_1(\boldsymbol{q}, \varphi) &= -\sin\varphi\, \boldsymbol{e}_1(\boldsymbol{q}) - \cos\varphi\, \boldsymbol{e}_2(\boldsymbol{q}), \\ \boldsymbol{n}_2(\boldsymbol{q}, \varphi) &= \cos\varphi\, \boldsymbol{e}_1(\boldsymbol{q}) - \sin\varphi\, \boldsymbol{e}_2(\boldsymbol{q}). \end{aligned} \tag{4.11}$$

Table 4.3
Particle variables $\boldsymbol{q}$, $v_\parallel$, $v_\perp$, φ.

Definition

$$\boldsymbol{q} = \boldsymbol{q}, \quad \boldsymbol{v} = v_\parallel \boldsymbol{b}(\boldsymbol{q}) + v_\perp \boldsymbol{n}_1(\boldsymbol{q}, \varphi).$$

Fundamental Lie brackets

$$[q_i, q_j] = 0, \qquad [\boldsymbol{q}, v_\parallel] = m^{-1}\boldsymbol{b}, \qquad [\boldsymbol{q}, v_\perp] = m^{-1}\boldsymbol{n}_1,$$

$$[\boldsymbol{q}, \varphi] = -\frac{1}{mv_\perp}\boldsymbol{n}_2, \qquad [v_\parallel, v_\perp] = \frac{1}{m}\boldsymbol{n}_2 \cdot \boldsymbol{D}, \qquad [v_\parallel, \varphi] = \frac{1}{mv_\perp}\boldsymbol{n}_1 \cdot \boldsymbol{D},$$

$$[v_\perp, \varphi] = -\frac{eB}{m^2 c v_\perp} - \frac{1}{mv_\perp}\boldsymbol{b} \cdot \boldsymbol{D}. \tag{1}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{v}) \to (\boldsymbol{q}, v_\parallel, v_\perp, \varphi)$

$$J = v_\perp. \tag{2}$$

Hamiltonian

$$H = (m/2)\left(v_\parallel^2 + v_\perp^2\right) + e\Phi(\boldsymbol{q}). \tag{3}$$

Equations of motion

$$\dot{\boldsymbol{q}} = v_\parallel \boldsymbol{b} + v_\perp \boldsymbol{n}_1, \tag{4}$$

$$\dot{v}_\parallel = v_\perp \boldsymbol{n}_2 \cdot \boldsymbol{D} + \frac{e}{m}\boldsymbol{b} \cdot \boldsymbol{E}, \tag{5}$$

$$\dot{v}_\perp = -v_\perp \boldsymbol{n}_2 \cdot \boldsymbol{D} + \frac{e}{m}\boldsymbol{n}_1 \cdot \boldsymbol{E}, \tag{6}$$

$$\dot{\varphi} = \frac{eB}{mc} + \boldsymbol{b} \cdot \boldsymbol{D} - \frac{v_\parallel}{v_\perp}\boldsymbol{n}_1 \cdot \boldsymbol{D} - \frac{e}{mv_\perp}\boldsymbol{n}_2 \cdot \boldsymbol{E}. \tag{7}$$

Remarks

All fields Φ, $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{b}$, $\boldsymbol{n}_i$ are evaluated at $\boldsymbol{x} = \boldsymbol{q}$; $\nabla \equiv \partial/\partial \boldsymbol{q}$,

$$\boldsymbol{D} \equiv \boldsymbol{D}(\boldsymbol{q}, v_\parallel, v_\perp, \varphi) = v_\parallel(\nabla \wedge \boldsymbol{b}) + v_\perp(\nabla \wedge \boldsymbol{n}_1). \tag{8}$$

We also note the following, very useful relations

$$\frac{\partial \boldsymbol{n}_1(\boldsymbol{q}, \varphi)}{\partial \varphi} = -\boldsymbol{n}_2(\boldsymbol{q}, \varphi), \qquad \frac{\partial \boldsymbol{n}_2(\boldsymbol{q}, \varphi)}{\partial \varphi} = \boldsymbol{n}_1(\boldsymbol{q}, \varphi). \tag{4.12}$$

We finally note that the construction described above leads to the following expression for the velocity of the particle:

$$\boldsymbol{v} = v_\parallel \boldsymbol{b}(\boldsymbol{q}) + v_\perp \boldsymbol{n}_1(\boldsymbol{q}, \varphi). \tag{4.13}$$

We now decide to use as phase space coordinates the variables

$$\boldsymbol{q},\ v_{\|},\ v_{\perp} \quad \text{and} \quad \varphi.$$

The angle φ is called the *gyrophase*. These variables are very similar to a set of cylindrical coordinates for the velocity; but one should not forget that the direction of the cylinder axis varies from one point to another.

Let us stress the following fact, which is an obvious property of these "quasi-cylindrical" coordinates. As $v_{\|}\boldsymbol{b}$ denotes the parallel velocity, which may actually be parallel or antiparallel to the magnetic field, $v_{\|}$ may be positive or negative:

$$v_{\|} \gtrless 0. \tag{4.14}$$

On the contrary, $v_{\perp}$ is the *absolute value* of the perpendicular velocity, and is necessarily non-negative,

$$v_{\perp} \geqslant 0. \tag{4.15}$$

We treat (4.13) as a pseudo-canonical transformation of the type (2.24): $(\boldsymbol{q},\ \boldsymbol{v}) \rightarrow (\boldsymbol{q},\ v_{\|},\ v_{\perp},\ \varphi)$ and go once more through the usual steps leading us to the results of table 4.3. The calculation of the fundamental Lie brackets is a little bit lengthy, but not difficult.

We have now derived a set of equations of motion for a set of physically important quantities. In spite of their compactness, these equations are still very complicated. We shall explore some simple cases in the next section. These will give us a hint for finding new, approximate transformations which simplify their solution. The three sets of variables considered in this section, i.e. $(\boldsymbol{q},\ \boldsymbol{p})$, $(\boldsymbol{q},\ \boldsymbol{v})$, $(\boldsymbol{q},\ v_{\|},\ v_{\perp},\ \varphi)$, will be called collectively *particle variables*, because in each of them half of the variables are the *coordinates* $\boldsymbol{q}$ *of the particle*.

1.5. Motion of a charged particle in simple electromagnetic fields

In order to gain insight into the problems of charged particle motion, we briefly consider the simplest cases. First, we assume that there is *no magnetic field*, $\boldsymbol{B} = 0$, and that *the electric field is homogeneous*, $\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}$. In this case, eqs. (5) and (6) in table 4.2 reduce to

$$\ddot{\boldsymbol{q}}(t) = (e/m)\boldsymbol{E}. \tag{5.1}$$

In this trivially simple case, the solution represents a uniformly accelerated motion of the particle, parallel or antiparallel to $\boldsymbol{E}$ (according to the sign of e). The solution is

$$\boldsymbol{q}(t) = \boldsymbol{q}^0 + \boldsymbol{v}^0 t + \tfrac{1}{2}(e/m)\boldsymbol{E}t^2, \tag{5.2}$$

where $\boldsymbol{q}^0$ and $\boldsymbol{v}^0$ are the initial position and velocity. The trajectory of the particle is, in general, a *parabola*.

Next, we consider $\boldsymbol{E} = 0$, and a *homogeneous magnetic field*, $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{B}$. In this case, the variables $\boldsymbol{q}$, $v_\parallel$, $v_\perp$, φ are particularly convenient. The definition of these variables is considerably simpler in the case of a homogeneous field. Indeed, the orientation of the vector $\boldsymbol{b}$ being constant, the local reference triad, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{b}$ of fig. 4.1 is everywhere parallel to the axes of the absolute reference frame (4.9). As a result, the transformation (4.13) no longer mixes the velocities and the positions; it reduces to

$$\boldsymbol{v} = v_\parallel \boldsymbol{b} + v_\perp \boldsymbol{n}_1(\varphi). \tag{5.3}$$

The vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{b}$ are independent of $\boldsymbol{q}$. The equations of motion (4)–(7) in table 4.3 reduce to

$$\dot{\boldsymbol{q}} = v_\parallel \boldsymbol{b} + v_\perp \boldsymbol{n}_1(\varphi), \qquad \dot{v}_\parallel = 0,$$

$$\dot{v}_\perp = 0, \qquad \dot{\varphi} = \Omega, \tag{5.4}$$

We introduced here the *Larmor frequency* Ω (sometimes called the cyclotron frequency),

$$\Omega = \frac{eB}{mc} \tag{5.5}$$

This quantity plays a basic role in plasma physics. Clearly, for a homogeneous magnetic field, the Larmor frequency is constant. Note that *Ω is positive or negative, according to the sign of the charge.*

Equations (5.4) have a number of simple properties; as a result, their solution is straightforward.

– *The length of the parallel and of the perpendicular velocity are separately constant*; thus

$$v_\parallel(t) = v_\parallel, \qquad v_\perp(t) = v_\perp. \tag{5.6}$$

– The last equation is *decoupled* from the others, because Ω is constant. Hence, *the gyrophase varies linearly in time*:

$$\varphi(t) = \varphi^0 + \Omega t. \tag{5.7}$$

– This result is substituted into the first equation, which is then easily integrated, using (4.12),

$$q_{\parallel}(t) = q_{\parallel}^0 + v_{\parallel} t, \tag{5.8}$$

$$\boldsymbol{q}_{\perp}(t) = \boldsymbol{q}_{\perp}^0 + \frac{v_{\perp}}{\Omega}\left[\boldsymbol{n}_2(\varphi^0 + \Omega t) - \boldsymbol{n}_2(\varphi^0)\right]. \tag{5.9}$$

The motion *parallel* to $\boldsymbol{B}$, determined by (5.8), is a uniform translation along the field lines, with constant velocity $v_{\parallel}$.

The function $\boldsymbol{q}_{\perp}(t)$ given by (5.9), is *strictly periodic in time* (remember eq. 4.11). Hence, eq. (5.9) describes the uniform rotation of the particle along a circular trajectory of radius ρ_L,

$$\rho_L = \frac{v_{\perp}}{|\Omega|} \tag{5.10}$$

centered at the point $\boldsymbol{Y}$,

$$\boldsymbol{Y} = \boldsymbol{q}^0 - \frac{v_{\perp}}{\Omega}\boldsymbol{n}_2(\boldsymbol{q}^0, \varphi^0) \tag{5.11}$$

This point, which depends only on the initial position and velocity of the particle (and not on time) is called the *guiding centre*. ρ_L is called the *Larmor radius*: it is proportional to the transverse velocity, and inversely proportional to the magnetic field intensity.

The combined motion takes place on a *helical trajectory* wound around a magnetic line of force. In this case, the dynamical variables $\boldsymbol{q}$, $v_{\parallel}$, $v_{\perp}$, φ have a particularly simple geometrical interpretation, shown in fig. 5.1.

Let us define another quantity, which will play an important role in forthcoming developments. Consider the projection of the trajectory on a plane perpendicular to $\boldsymbol{B}$ (say, the plane $x_3 = 0$). This circle can be considered formally as a current-carrying loop. The average current along the trajectory can be defined as the amount of charge traversing per unit time an area Q perpendicular to the trajectory (fig. 5.1):

$$I = e\frac{\Omega}{2\pi}.$$

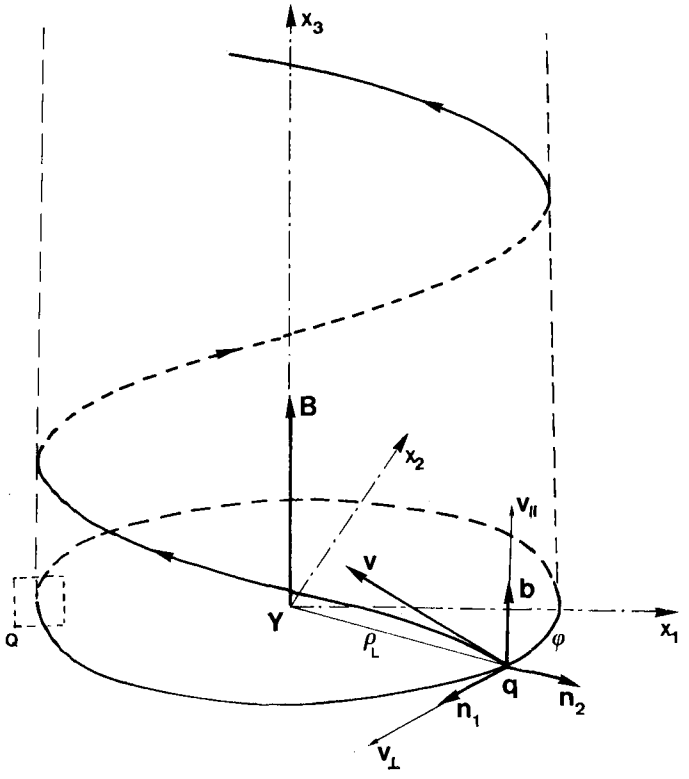

Fig. 5.1. Motion of a positively charged particle in a constant magnetic field. Note the absolute reference frame (x_1, x_2, x_3), the moving local triad $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{b})$, the guiding centre $\boldsymbol{Y}$, the gyrophase φ, and the components $v_{\parallel}$, $v_{\perp}$ of $\boldsymbol{v}$.

As is well known, a current loop can be characterized by an associated *magnetic moment* μ, of magnitude μ,

$$\mu = \frac{1}{c} IS = \frac{1}{c} \frac{e\Omega}{2\pi} \pi \rho_L^2,$$

where S is the area of the loop. Using eqs. (5.5) and (5.10), we get

$$\mu = \frac{m v_\perp^2}{2B}. \tag{5.12}$$

When considered as a vector, this magnetic moment points in the direction opposite to $\boldsymbol{B}$, $\boldsymbol{\mu} = -\mu \boldsymbol{b}$. The Larmor loop can be considered as *diamagnetic*.

Although these considerations are somewhat formal, it will be seen that the *magnetic moment* μ associated with the Larmor gyration plays a major role in the forthcoming developments.

We now consider the motion of a particle in the simultaneous presence of a magnetic field $\boldsymbol{B}$ and an electric field $\boldsymbol{E}$, *both homogeneous*.

The combined equations of motion (5) and (6) in table 4.2 yield

$$\ddot{\boldsymbol{q}} = \frac{e}{mc}(\boldsymbol{v} \wedge \boldsymbol{B}) + \frac{e}{m}\boldsymbol{E}. \tag{5.13}$$

The component of $\boldsymbol{q}$ parallel to the magnetic field obeys

$$\boldsymbol{q} = \boldsymbol{y} + \boldsymbol{w}_E t, \tag{5.15}$$

The perpendicular component obeys (5.13) with $\boldsymbol{E}_\perp$ replacing $\boldsymbol{E}$. In the case of uniform fields, it is still possible to separate neatly the effects of the electric and the magnetic fields. To this purpose, we make the change of variables $\boldsymbol{q} \rightarrow \boldsymbol{y}$, which is a simple Galilean transformation to a frame moving with a constant velocity $\boldsymbol{w}_E$ (Sivukhin 1965),

$$\boldsymbol{q} = \boldsymbol{y} + \boldsymbol{w}_E t, \tag{5.15}$$

where $\boldsymbol{w}_E$ is taken perpendicular to the magnetic field,

$$\boldsymbol{w}_E \cdot \boldsymbol{B} = 0. \tag{5.16}$$

The perpendicular equation of motion becomes

$$\ddot{\boldsymbol{y}}_\perp = \frac{e}{mc}(\dot{\boldsymbol{y}} \wedge \boldsymbol{B}) + \frac{e}{mc}(\boldsymbol{w}_E \wedge \boldsymbol{B}) + \frac{e}{m}\boldsymbol{E}_\perp. \tag{5.17}$$

Clearly, if we choose $\boldsymbol{w}_E$ such that

$$c^{-1}(\boldsymbol{w}_E \wedge \boldsymbol{B}) + \boldsymbol{E}_\perp = 0 \tag{5.18}$$

then (5.17) reduces to the equation of a particle in a magnetic field alone, i.e. to the problem solved above. It follows that the complete motion (in the initial reference frame) results from the superposition of the three "component" motions,

$$\boldsymbol{q}(t) = q_\parallel(t)\boldsymbol{b} + \boldsymbol{y}_\perp(t) + \boldsymbol{w}_E t. \tag{5.19}$$

The first term represents a uniformly accelerated motion in the direction of the magnetic field (see 5.2),

$$q_{\parallel}(t) = q_{\parallel}^0 + v_{\parallel}^0 t + \tfrac{1}{2}(e/m)E_{\parallel}t^2. \tag{5.20}$$

The second term represents the Larmor gyration, described previously, around a fixed guiding centre defined by (5.11),

$$\boldsymbol{y}(t) = \boldsymbol{Y} + \frac{v_{\perp}^0}{\Omega}\boldsymbol{n}_2(\varphi^0 + \Omega t). \tag{5.21}$$

The third term represents the most "original" kind of motion appearing in this problem; it is a uniform motion directed perpendicularly to both the electric and the magnetic fields, with a constant velocity $\boldsymbol{w}_{\mathrm{E}}$, derived from (5.18) and (5.16),

$$\boldsymbol{w}_E = \frac{c}{B^2}(\boldsymbol{E} \wedge \boldsymbol{B}). \tag{5.22}$$

This is called a *drift motion* and $\boldsymbol{w}_E$ is the *electric drift velocity*. Let us analyze in detail this important type of motion in the simple case where $q_{\parallel}^0 = 0$, $v_{\parallel}^0 = 0$ and $E_{\parallel} = 0$, i.e. when the motion occurs in the plane $q_3 = 0$ (fig. 5.2). The combination of the last two terms in (5.19) can be pictured as a *distorted Larmor gyration around a guiding centre moving at velocity* $\boldsymbol{w}_E$; the resulting trajectory is a trochoid (or oblate cycloid). Its origin can be easily understood in physical terms.

Consider a positively charged particle. In the absence of $\boldsymbol{E}$, it gyrates around a fixed centre in the direction *abcd* (fig. 5.2A). We now switch on the field $\boldsymbol{E}$ and consider a particle starting at *a*. As it moves downward under the action of $\boldsymbol{B}$, it is slowed down by $\boldsymbol{E}$. The resulting distortion of the trajectory can be understood as a gradual decrease of the instantaneous Larmor radius (5.10), which reaches a minimum at *b*. On the second half of the journey, the particle is pushed upwards by $\boldsymbol{B}$ and is accelerated by $\boldsymbol{E}$, hence the local radius of curvature increases back to its initial value. After a complete cycle, the position of the particle is shifted to the right: this is the origin of the electric drift.

A very remarkable feature of the electric drift is the fact that $\boldsymbol{w}_E$ does not depend on the charge of the particle. This paradox of an electric field acting in the same way on positive and negative particles is, of course, only apparent. The same qualitative analysis, applied to a negatively charged particle (which gyrates in the opposite sense!), explains the effect (fig. 5.2B). We thus emphasize the following important feature: *Under the action of an electric field, the negative and positive charges drift in the same direction.*

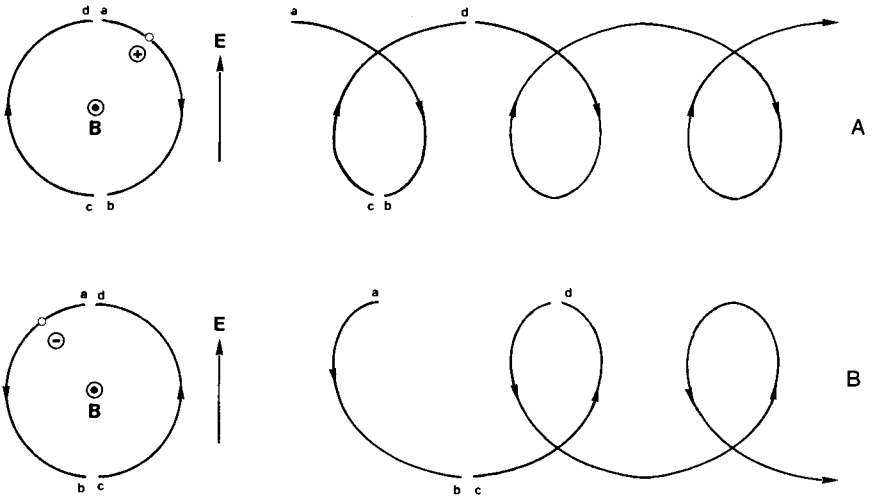

Fig. 5.2. Physical origin of the electric drift. (A) Positively charged particle, (B) negatively charged particle.

We conclude this analysis with a word of caution. Equation (5.19) was obtained by a correct solution of the equations of motion (5.13) and may be considered as the exact solution of the problem, for any values of the fields $\boldsymbol{B}$ and $\boldsymbol{E}$. Nevertheless, it leads to unbearable physical paradoxes. The most obvious is the following. We may try to naively take the limit $B \to 0$, for fixed E. The equation of motion (5.13) smoothly goes over into eq. (5.1); but the *solution* (5.19) does *not* go over into (5.2). In particular, in the limit $B \to 0$, the drift velocity (5.22) becomes infinite! This very fact points towards the explanation of the paradox: somewhere in the limiting process, we have crossed the barrier of validity of classical mechanics and entered the realm of relativistic mechanics, in which the equations of motion (5.13) are no longer valid. A precise criterion of applicability of the equations is easily found. We must guarantee that the absolute value of the drift velocity is much smaller than the speed of light:

$$w_E = \frac{E_\perp B}{B^2} c \ll c$$

which requires $E_\perp \ll B$. We also have a criterion involving the parallel electric field. Indeed, the increment in velocity over a Larmor period must also be much smaller than c. From (5.20) we find

$$\Delta v_\parallel = \frac{e}{m} E_\parallel \frac{1}{\Omega} = c \frac{E_\parallel}{B} \ll c$$

which requires $E_{\|} \ll B$. Hence, in order for the equations (5.13) to possess solutions which do not transgress their own limits of validity, i.e. classical mechanics, we need the following condition on the absolute values of the electric and magnetic fields:

$$E \ll B. \tag{5.23}$$

Whenever this condition is not satisfied, one needs to use relativistic equations of motion for treating the problem (Sivukhin 1965).

This discussion shows how "wild" the motion of a single charged particle can be, even under the simplest imaginable conditions. Indeed, even constant fields of infinitesimally small intensity can produce relativistically large velocities, provided $E \approx B$.

We now go over to the study of the more realistic situations where the fields $\boldsymbol{B}(\boldsymbol{x})$ and $\boldsymbol{E}(\boldsymbol{x})$ are *inhomogeneous*. In this case, the equations of motion (4)–(7) in table 4.3 become inextricably complicated, even in the simpler case when $\boldsymbol{E}(\boldsymbol{x}) = 0$. Indeed, in the homogeneous case (5.4), the gyrophase φ obeys an equation decoupled from all the other variables; it can be solved separately, and its simple solution is used in order to solve the remaining equations. In the general case, the right-hand sides of all the equations depend on the gyrophase (through the vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{D}$) and represent therefore oscillating functions. Such equations present considerable difficulties (see the discussion in section 1.6).

The results of the present section suggest the study of some situations which are physically interesting and also mathematically tractable. Consider, for instance, a case where $\boldsymbol{E} = 0$, and the magnetic field is inhomogeneous, but straight. Moreover, let its intensity depend on a single coordinate,

$$\boldsymbol{B}(\boldsymbol{x}) = B(x_1)\, \boldsymbol{b}.$$

In this case, the motion in the x_1, x_2 plane can be understood by the qualitative argument described above (fig. 5.3). As a positively charged particle starts on its downward trip, it enters a region of higher field intensity. If the variation of B is sufficiently slow, the trajectory will be only slightly distorted, in a way similar to fig. 5.2: the Larmor radius (5.10) decreases along the downward journey and increases along the upward one. The result is again a *drift motion*, directed perpendicularly to both $\boldsymbol{B}$ and ∇B,

$$\boldsymbol{w}_{\nabla B} \sim \boldsymbol{B} \wedge (\nabla B). \tag{5.24}$$

Applying the same argument to a negatively charged particle we find (fig. 5.3B) that the resulting drift is in the opposite direction. Thus, *a gradient of*

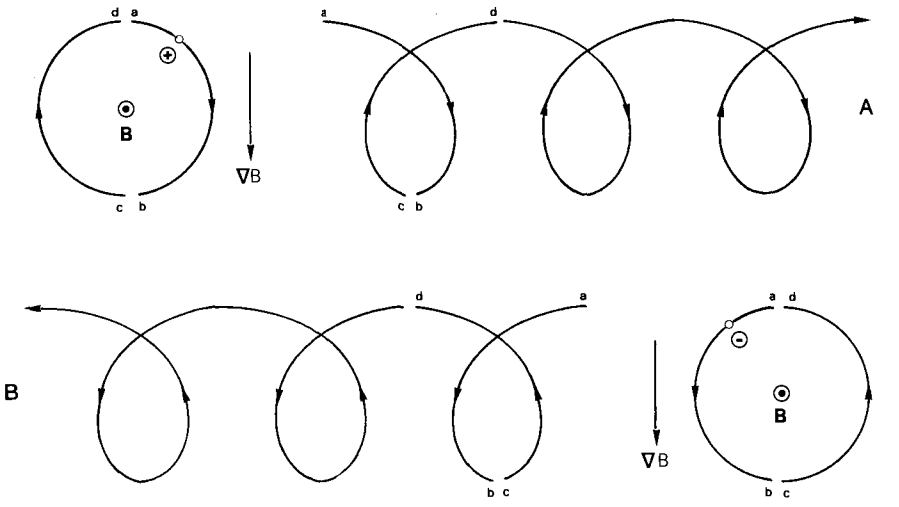

Fig. 5.3. Drift due to the gradient of magnetic field intensity. (A) positively charged particle, (B) negatively charged particle.

magnetic field intensity causes oppositely charged particles to drift in opposite directions. We do not attempt to derive a precise expression of this drift velocity at this stage. We also anticipate that this is not the only effect of the field inhomogeneity: a systematic discussion will be given in the forthcoming sections. This example only suggests that *if the inhomogeneity is sufficiently weak, its effect is a slow drift of the instantaneous guiding centre of the particle*. A *weak inhomogeneity* means, more precisely, that the scale length defined by the magnetic field inhomogeneity

$$L_{\mathrm{H}}^{-1} = \frac{1}{B} |\nabla B| \tag{5.25}$$

is, on the average, much larger than the Larmor radius ρ_{L} (5.10),

$$\rho_{\mathrm{L}} \ll L_{\mathrm{H}}. \tag{5.26}$$

Clearly, in the general case of an inhomogeneous magnetic field, we require a more complete definition of a weak inhomogeneity, by asking that *all* the characteristic lengths of the magnetic field be large, compared to the Larmor radius. These characteristic lengths include the radius of curvature ρ (3.18) and the radius of torsion (3.21).

Fig. 5.4. Exact (a) and smoothed (b) trajectories in a weakly inhomogeneous field.

One may expect, qualitatively, that in a general weakly inhomogeneous situation, a typical particle trajectory looks schematically as in fig. 5.4a. The particle gyrates very rapidly (with frequency $\Omega \sim B$) around a guiding centre, which drifts slowly through space along a trajectory having none of the sharp curbs of the Larmor gyration. It may happen that, *for some problems* (not all!), we may disregard the rapid gyrations of the particles and be interested only in the large-scale, smooth trajectory shown in fig. 5.4b. The particle motion is then assimilated with its guiding centre motion: only if we "look with a microscope" shall we see the fine structure of fig. 5.4a. This smoothed picture is called the *drift approximation*. One may expect that, whenever it is valid, it leads to a much simpler description of the motion. However, the precise definition and use of this approximation is a subtle matter, which has been a subject of research even in very recent years. It will be discussed in the forthcoming sections.

1.6. The drift approximation: The method of the average

We now introduce a mathematical formulation of the drift approximation, discussed qualitatively in the previous section. Although the simpler aspects are clear and can be found in all elementary textbooks on plasma physics, the precise formulation of the problem is a difficult and subtle matter.

We have seen that the idea of the drift approximation applies whenever the Larmor radius is much smaller than the spatial scale of variation of the external fields (5.25). Let us compare, for instance, the size of the first two terms of eq. (7) in table 4.3 [recalling definition (8)],

$$\left|\frac{eB}{mc}\right| : |\boldsymbol{b}\cdot\boldsymbol{D}| \sim \frac{v_\perp}{\rho_L} : \left(\frac{v_\parallel}{L_H} + \frac{v_\perp}{L'_H}\right),$$

where L_H and L'_H are both characteristic lengths measuring the spatial scale of the field and are introduced by $\nabla \wedge \boldsymbol{b}$ and by $\nabla \wedge \boldsymbol{n}_1$, respectively. When the drift approximation applies ($\rho_L \ll L_H, L'_H$), the first term clearly

dominates. This fact can be expressed by a simple mathematical trick. A *scaling parameter* is introduced by the replacement

$$\frac{eB}{mc} \equiv \Omega \rightarrow \frac{1}{\epsilon}\Omega \tag{6.1}$$

in the dynamical equation of motion for $\dot{\varphi}$, eq. (7) in table 4.3. It is easily checked that the resulting equations still derive from the Hamiltonian (3), provided we make the same replacement in the Lie bracket $[v_\perp, \varphi]$ of eq. (1). The resulting set of equations is recopied in table 6.1 for easy reference. The scaling parameter ϵ is a simple indicator of the order of magnitude of the various terms.

The *drift approximation* is obtained, by definition, when ϵ is considered as a small parameter,

$$\epsilon \ll 1. \tag{6.2}$$

Let us stress the fact that, at the end of the calculations, the formal parameter is given the value $\epsilon = 1$, in order to restore the true physical quantities.

It is easily seen that the equations of motion (5)–(8) in table 6.1 can be written collectively in the form

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = f_k(x_i, \varphi), \tag{6.3}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{\epsilon}\Omega(x_i) + a(x_i, \varphi), \tag{6.4}$$

where $f_k(x_i, \varphi)$ and $a(x_i, \varphi)$ are *periodic functions of* φ.

Qualitatively, it may be said that, if ϵ is small enough, the motion resembles the one pictured in figs. 5.2 and 5.3. The motion is *nearly periodic*, in the sense that when the particle completes a gyration cycle, it "misses" its starting point by a small amount; as a result, the guiding centre *drifts* slowly through space.

Thus, eq. (6.4) describes, to dominant order in ϵ, a very rapid temporal variation of φ. But the more refined description is very complicated, because of the dependence of the coefficients on the gyrophase. In order to grasp the difficulty, we shall simply quote a passage from Kruskal's (1962) basic paper, whose clarity cannot be matched: *"The gyration it depicts may be very uneven, full of sudden accelerations and decelerations, since $a(x_i, \varphi)$ may fluctuate wildly; further, and in fact, more serious, the net rate of drift is not at all in evidence, since in the course of gyration $\dot{q}$ may point in all directions, even quite opposite to the direction that y is effectively drifting in"*.

Table 6.1
Scaled particle variables $\boldsymbol{q}$, $v_{\|}$, $v_{\perp}$, φ.

Definition

$$\boldsymbol{q} = \boldsymbol{q}, \quad \boldsymbol{v} = v_{\|}\boldsymbol{b}(\boldsymbol{q}) + v_{\perp}\boldsymbol{n}_1(\boldsymbol{q}, \varphi).$$

Fundamental Lie brackets

$$[q_i, q_j] = 0 \qquad [\boldsymbol{q}, v_{\|}] = m^{-1}\boldsymbol{b}, \qquad [\boldsymbol{q}, v_{\perp}] = m^{-1}\boldsymbol{n}_1,$$

$$[\boldsymbol{q}, \varphi] = -\frac{1}{mv_{\perp}}\boldsymbol{n}_2, \qquad [v_{\|}, v_{\perp}] = m^{-1}\boldsymbol{n}_2 \cdot \boldsymbol{D}, \qquad [v_{\|}, \varphi] = \frac{1}{mv_{\perp}}\boldsymbol{n}_1 \cdot \boldsymbol{D},$$

$$[v_{\perp}, \varphi] = -\frac{1}{\epsilon}\frac{\Omega}{mv_{\perp}} - \frac{1}{mv_{\perp}}\boldsymbol{b} \cdot \boldsymbol{D}. \tag{1}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{v}) \rightarrow (\boldsymbol{q}, v_{\|}, v_{\perp}, \varphi)$

$$J = v_{\perp}. \tag{2}$$

Hamiltonian

$$H = (m/2)\left(v_{\|}^2 + v_{\perp}^2\right) + e\Phi. \tag{3}$$

Equations of motion

$$\dot{\boldsymbol{q}} = v_{\|}\boldsymbol{b} + v_{\perp}\boldsymbol{n}_1, \tag{4}$$

$$\dot{v}_{\|} = v_{\perp}\boldsymbol{n}_2 \cdot \boldsymbol{D} + \frac{e}{m}\boldsymbol{b} \cdot \boldsymbol{E}, \tag{5}$$

$$\dot{v}_{\perp} = -v_{\|}\boldsymbol{n}_2 \cdot \boldsymbol{D} + \frac{e}{m}\boldsymbol{n}_1 \cdot \boldsymbol{E}, \tag{6}$$

$$\dot{\varphi} = \frac{1}{\epsilon}\Omega + \boldsymbol{b} \cdot \boldsymbol{D} - \frac{v_{\|}}{v_{\perp}}\boldsymbol{n}_1 \cdot \boldsymbol{D} - \frac{e}{mv_{\perp}}\boldsymbol{n}_2 \cdot \boldsymbol{E}. \tag{7}$$

Remarks

All fields Φ, $\boldsymbol{E}$, B, Ω, $\boldsymbol{b}$, $\boldsymbol{n}_i$ are evaluated at $\boldsymbol{x} = \boldsymbol{q}$; $\nabla \equiv \partial/\partial \boldsymbol{q}$,

$$\boldsymbol{D} = v_{\|}(\nabla \wedge \boldsymbol{b}) + v_{\perp}(\nabla \wedge \boldsymbol{n}_1). \tag{8}$$

All these difficulties are due to the dependence of the coefficients on the gyrophase. In order to cure these problems, we may try to realize a program, which is most clearly stated by Kruskal (1962), whom we quote again: "*We therefore ask whether we can find an infinitely differentiable formal transformation to new variables which are similar to* x_i *and* φ *and satisfy equations similar to our eqs. (6.3), (6.4), but without the odious dependence on the angle-like variable – and not only to lowest order, but to all orders". We shall see that the*

answer is yes". The realization of this program has been the object of a large amount of work.

Most treatments of the charged particle problem up to 1979 (e.g. Kruskal 1962, Northrop 1963, Gardner 1966, Northrop et al. 1966, Northrop and Rome 1978, Hastie et al. 1967, Baños 1967, Sivukhin 1965, Morozov and Soloviev 1966) are variations on the theme of the *method of the average*. The idea of this method can be traced as far back as 1937 in the works by Krylov and Bogoliubov on the asymptotic solutions of non-linear differential systems (English transl.: Krylov and Bogoliubov (1947); see also the book by Bogoliubov and Mitropolsky (1962)). The idea of the method is clearly expressed in the paper by Morozov and Soloviev (1966).

We replace the variables (x_k, φ) by a new set of "*averaged variables*" (ξ_k, ϕ) *differing little from* (x_k, φ) *and satisfying differential equations which do not contain the rapidly varying phase*. We thus try to find a transformation

$$x_k = \xi_k + \epsilon g_{1k}(\xi_i, \phi) + \epsilon^2 g_{2k}(\xi_i, \phi) + \cdots, \tag{6.5}$$

$$\varphi = \phi + \epsilon q_1(\xi_i, \phi) + \epsilon^2 q_2(\xi_i, \phi) + \cdots, \tag{6.6}$$

such that

$$\frac{\mathrm{d}\xi_k}{\mathrm{d}t} = F_{0k}(\xi_i) + \epsilon F_{1k}(\xi_i) + \epsilon^2 F_{2k}(\xi_i) + \cdots, \tag{6.7}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{\epsilon}\Omega(\xi_i) + \omega_0(\xi_i) + \epsilon\omega_1(\xi_i) + \cdots. \tag{6.8}$$

We illustrate the method of construction of the new equations only in the simplest case, i.e. to zeroth order. Using eq. (6.5), we write

$$\dot{x}_k = \dot{\xi}_k + \epsilon \frac{\partial g_{1k}}{\partial \xi_i}\dot{\xi}_i + \epsilon \frac{\partial g_{1k}}{\partial \phi}\dot{\phi} + \mathrm{O}(\epsilon^2)$$

Using now eqs. (6.7), (6.8) and noting that $\dot{\phi}$ has a term in ϵ^{-1}, we have

$$\dot{x}_k = F_{0k} + \frac{\partial g_{1k}}{\partial \phi}\Omega + \mathrm{O}(\epsilon). \tag{6.9}$$

On the other hand, (6.3) yields

$$\dot{x}_k = f_k(x_i, \varphi) = f_k(\xi_i + \epsilon g_{1i} + \cdots, \phi + \epsilon q_1 + \cdots) = f_k(\xi_i, \phi) + \mathrm{O}(\epsilon). \tag{6.10}$$

Comparing the zeroth order terms in (6.9) and (6.10), we get

$$F_{0k}(\xi_i) + \frac{\partial g_{1k}(\xi_i, \phi)}{\partial \phi} \Omega(\xi_i) = f_k(\xi_i, \phi). \tag{6.11}$$

We are still faced with the problem of suppressing the oscillations. To this purpose we introduce an *averaging operation*, denoted by an overbar, applicable to any periodic function of ϕ:

$$\bar{G}(\xi_i) \equiv \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \, G(\xi_i, \phi). \tag{6.12}$$

By means of this operation, any periodic function can be represented as a sum of an *average term* $\bar{G}$ and of an *oscillating term* $\tilde{G}$, whose average is zero,

$$G(\xi_i, \phi) = \bar{G}(\xi_i) + \tilde{G}(\xi_i, \phi). \tag{6.13}$$

Returning to eq. (6.11), we note that there is still an element of arbitrariness, which is suppressed by making a *choice*:

$$\bar{g}_{1k}(\xi_i) = 0, \qquad \bar{q}_1(\xi_i) = 0.$$

We then obtain from (6.5) and (6.6),

$$\xi_k = \bar{x}_k + \mathrm{O}(\epsilon^2), \qquad \phi = \bar{\varphi} + \mathrm{O}(\epsilon^2), \tag{6.14}$$

together with the approximate equations of motion

$$\dot{\bar{\xi}}_k = \bar{f}_k(\xi_i), \qquad \dot{\bar{\phi}} = \frac{1}{\epsilon} \Omega(\xi_i). \tag{6.15}$$

Clearly, the algorithm can be pursued systematically (Sivukhin 1965, Morozov and Soloviev 1966), but we shall not go further here. We rather apply the method to our specific problem, in order to bring out some interesting and useful features.

For physical reasons suggested by the qualitative discussion in section 1.5, the equation of motion for $\dot{\boldsymbol{q}}$ is not expected to be simple, even after averaging. All authors start by introducing ad hoc an instantaneous guiding centre, whose coordinates are denoted here by $\boldsymbol{y}$ (rather than $\boldsymbol{Y}$ as in section 1.5) (see eq. 5.11),

$$\boldsymbol{y} = \boldsymbol{q} - \epsilon \frac{v_\perp}{\Omega(\boldsymbol{q})} \boldsymbol{n}_2(\boldsymbol{q}, \varphi). \tag{6.16}$$

The *exact* equation of motion for y is easily obtained by applying the Hamiltonian formalism in the variables of table 6.1:

$$\dot{y} = \left[q - \epsilon \frac{v_\perp}{\Omega(q)} n_2(q, \varphi),\ H \right]$$

$$= v_\parallel b + \frac{\epsilon}{\Omega} \Big(v_\parallel v_\perp \frac{1}{B} n_2 (b \cdot \nabla) B + v_\perp^2 \frac{1}{B} n_2 (n_1 \cdot \nabla) B$$

$$+ v_\parallel^2 b \wedge [(b \cdot \nabla) b]$$

$$+ v_\parallel v_\perp \{ b \wedge [(n_1 \cdot \nabla) b] + n_1 \wedge [(b \cdot \nabla) b] \}$$

$$+ v_\perp^2 n_1 \wedge [(n_1 \cdot \nabla) b] + v_\parallel v_\perp (n_2 \cdot \nabla) b \Big) + \epsilon \frac{c}{B} E \wedge B. \tag{6.17}$$

All the fields n_1, n_2, b, B, Ω and E on the right-hand side of the above equations are evaluated at the particle's position $x = q$. We apply to this complicated equation the method of the average, introducing the *average guiding centre position* *

$$Y = \bar{y}, \tag{6.18}$$

whose equation of motion is obtained by averaging the right-hand side of (6.17), according to (6.15). We note that the oscillating terms in this equation are all of the form of products involving one or two of the unit vectors $n_j(q, \varphi)$. In order to calculate the average, we derive a set of interesting and very useful properties of such expressions.

We recall that the unit vectors n_1, n_2, b form a complete right-handed triad (4.10) at each point q. Therefore, they satisfy the closure relation

$$n_1 n_1 + n_2 n_2 + bb = I, \tag{6.19}$$

* At this point we see that the idea of "average variable" is ambiguous. y is considered as a dynamical variable independent of the others, in particular of the gyrophase φ; its average according to (6.12) is therefore identical to itself! The correct interpretation of (6.18) is: "Y is the variable obeying the equation of motion (6.17), averaged over the gyrophase". Actually, the average of y, in the context of the differential equations (6.3), should rather be interpreted as a time-average (this is, indeed, the point of view of Sivukhin 1965). Its realization through a gyrophase-average (6.12) is analogous to the ergodic hypothesis of statistical mechanics.

where $\boldsymbol{I}$ is the unit tensor. The vectors $\boldsymbol{n}_j$ are linear combinations of sin φ and cos φ, obeying eqs. (4.1), which also imply

$$\frac{\partial^2}{\partial\varphi^2}\boldsymbol{n}_j = -\boldsymbol{n}_j, \quad j = 1, 2. \tag{6.20}$$

As a result, the average (6.12) of any one of these vectors is zero:

$$\overline{\boldsymbol{n}_j(\boldsymbol{q}, \varphi)} = 0. \tag{6.21}$$

Likewise, the average of a product of any odd number of factors $\boldsymbol{n}_j$ is zero. For products of even numbers of factors $\boldsymbol{n}_j$, the situation is more complicated; the calculation of their averages is made easy by an elegant formalism due to Littlejohn (1981). Consider the four tensors

$$\boldsymbol{P}_0 = \boldsymbol{n}_1\boldsymbol{n}_1 + \boldsymbol{n}_2\boldsymbol{n}_2, \qquad \boldsymbol{P}_1 = \boldsymbol{n}_2\boldsymbol{n}_1 - \boldsymbol{n}_1\boldsymbol{n}_2,$$

$$\boldsymbol{Q}_0 = \boldsymbol{n}_1\boldsymbol{n}_2 + \boldsymbol{n}_2\boldsymbol{n}_1, \qquad \boldsymbol{Q}_1 = \boldsymbol{n}_1\boldsymbol{n}_1 - \boldsymbol{n}_2\boldsymbol{n}_2. \tag{6.22}$$

Using eqs. (4.12) we obtain

$$\frac{\partial}{\partial\varphi}\boldsymbol{P}_m = 0,$$

$$\frac{\partial^2}{\partial\varphi^2}\boldsymbol{Q}_m = -4\boldsymbol{Q}_m, \quad m = 0, 1. \tag{6.23}$$

Thus, the two tensors $\boldsymbol{P}_m$ are independent of φ, whereas the two tensors $\boldsymbol{Q}_m$ are linear combinations of sin 2φ, cos 2φ (second harmonics). We thus have

$$\overline{\boldsymbol{P}}_0 = \boldsymbol{P}_0, \qquad \overline{\boldsymbol{P}}_1 = \boldsymbol{P}_1,$$

$$\overline{\boldsymbol{Q}}_0 = 0, \qquad \overline{\boldsymbol{Q}}_1 = 0. \tag{6.24}$$

We note that eqs. (6.22) can be inverted,

$$\boldsymbol{n}_1\boldsymbol{n}_1 = \tfrac{1}{2}(\boldsymbol{P}_0 + \boldsymbol{Q}_1), \qquad \boldsymbol{n}_2\boldsymbol{n}_2 = \tfrac{1}{2}(\boldsymbol{P}_0 - \boldsymbol{Q}_1),$$

$$\boldsymbol{n}_1\boldsymbol{n}_2 = \tfrac{1}{2}(-\boldsymbol{P}_1 + \boldsymbol{Q}_0), \qquad \boldsymbol{n}_2\boldsymbol{n}_1 = \tfrac{1}{2}(\boldsymbol{P}_1 + \boldsymbol{Q}_0). \tag{6.25}$$

By means of these relations, any tensor, bilinear in $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, can be uniquely decomposed into an average part and an oscillating part, as in (6.13): the averaging of such a tensor is thus elementary.

In order to complete the picture, we derive several additional useful identities. From (6.22) and (6.19), we immediately get

$$\boldsymbol{P}_0 \cdot \boldsymbol{A} = \boldsymbol{A} - (\boldsymbol{b} \cdot \boldsymbol{A})\boldsymbol{b}, \tag{6.26}$$

where $\boldsymbol{A}$ is any vector independent of φ. We now calculate

$$\begin{aligned}\boldsymbol{b} \wedge \boldsymbol{A} &= (\boldsymbol{n}_1\boldsymbol{n}_1 + \boldsymbol{n}_2\boldsymbol{n}_2 + \boldsymbol{b}\boldsymbol{b}) \cdot (\boldsymbol{b} \wedge \boldsymbol{A}) \\ &= \boldsymbol{n}_1[\boldsymbol{n}_1 \cdot (\boldsymbol{b} \wedge \boldsymbol{A})] + \boldsymbol{n}_2[\boldsymbol{n}_2 \cdot (\boldsymbol{b} \wedge \boldsymbol{A})] \\ &= \boldsymbol{n}_1[\boldsymbol{A} \cdot (\boldsymbol{n}_1 \wedge \boldsymbol{b})] + \boldsymbol{n}_2[\boldsymbol{A} \cdot (\boldsymbol{n}_2 \wedge \boldsymbol{b})] \\ &= -\boldsymbol{n}_1\boldsymbol{n}_2 \cdot \boldsymbol{A} + \boldsymbol{n}_2\boldsymbol{n}_1 \cdot \boldsymbol{A} = \boldsymbol{P}_1 \cdot \boldsymbol{A},\end{aligned}$$

where we made use of (6.19), (4.10) and of some vector product identities. Thus

$$\boldsymbol{P}_1 \cdot \boldsymbol{A} = \boldsymbol{b} \wedge \boldsymbol{A}. \tag{6.27}$$

Similar identities are

$$\boldsymbol{P}_0 : \nabla \boldsymbol{A} = \nabla \cdot \boldsymbol{A} - \boldsymbol{b} \cdot \nabla \boldsymbol{A} \cdot \boldsymbol{b}, \tag{6.28}$$

$$\boldsymbol{P}_1 : \nabla \boldsymbol{A} = \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{A}). \tag{6.29}$$

On the right-hand side of (6.28) we use an abbreviated notation which is very convenient in suppressing cumbersome parentheses: we convene that *the operator* ∇ *acts only on the factor written next to it* (i.e. $\boldsymbol{A}$).

To complete the list, we derive some properties of expressions involving *gradients* of the vectors $\boldsymbol{n}_j$. These follow from the fact that the latter are orthogonal unit vectors,

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_1 = \boldsymbol{n}_2 \cdot \boldsymbol{n}_2 = 1, \qquad \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0.$$

Hence (adding similar identities for $\boldsymbol{b}$)

$$\nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_1 = \nabla \boldsymbol{n}_2 \cdot \boldsymbol{n}_2 = \nabla \boldsymbol{b} \cdot \boldsymbol{b} = 0 \tag{6.30}$$

and

$$\nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = -\nabla \boldsymbol{n}_2 \cdot \boldsymbol{n}_1. \tag{6.31}$$

From (4.11) also follows that

$$\boldsymbol{R} \equiv \nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \nabla \boldsymbol{e}_1 \cdot \boldsymbol{e}_2. \tag{6.32}$$

The vector $\boldsymbol{R}$ (which will appear later again) is thus *independent of the gyrophase* *.Two additional identities are useful:

$$\nabla \cdot \boldsymbol{P}_0 = -\boldsymbol{b}(\nabla \cdot \boldsymbol{b}) - (\boldsymbol{b} \cdot \nabla)\boldsymbol{b}, \tag{6.33}$$

$$\nabla \cdot \boldsymbol{P}_1 = \nabla \wedge \boldsymbol{b}. \tag{6.34}$$

We now come back to our initial problem: the averaging of eq. (6.17). As a first step, we must express all the fields ($\boldsymbol{b}$, $\boldsymbol{n}_{\mathrm{m}}$, ,...,) on the right-hand side of this equation in terms of the guiding centre position. Thus

$$\boldsymbol{b}(\boldsymbol{q}) = \boldsymbol{b}(\boldsymbol{Y}) + \epsilon \frac{v_\perp}{\Omega(\boldsymbol{Y})} \boldsymbol{n}_2(\boldsymbol{Y}, \phi) \cdot \nabla \boldsymbol{b}(\boldsymbol{Y}) + \mathrm{O}(\epsilon^2) \tag{6.35}$$

and similar expressions for the other fields. In the terms of (6.17), containing ϵ, we may forget this correction, but in the zeroth order term we must express $v_\parallel \boldsymbol{b}(\boldsymbol{q})$ according to (6.35). We are now ready for the averaging. Eliminating right away the terms linear in $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, we obtain

$$\dot{\boldsymbol{Y}} = v_\parallel \boldsymbol{b} + \epsilon \frac{c}{B^2}(\boldsymbol{E} \wedge \boldsymbol{B}) + \frac{\epsilon}{\Omega}\left(v_\parallel^2 \boldsymbol{b} \wedge (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + v_\perp^2 \frac{1}{B} \overline{\boldsymbol{n}_2 \boldsymbol{n}_1 \cdot \nabla \boldsymbol{B}} + v_\perp^2 \overline{\boldsymbol{n}_1 \wedge (\boldsymbol{n}_1 \cdot \nabla)\boldsymbol{b}} \right), \tag{6.36}$$

where all fields are evaluated at $\boldsymbol{x} = \boldsymbol{Y}$ and $\nabla \equiv \partial / \partial \boldsymbol{Y}$. The first average is easily calculated:

$$\overline{\boldsymbol{n}_2 \boldsymbol{n}_1 \cdot \nabla \boldsymbol{B}} = \tfrac{1}{2}\left(\overline{\boldsymbol{P}}_1 + \overline{\boldsymbol{Q}}_0\right) \cdot \nabla \boldsymbol{B} = \tfrac{1}{2} \boldsymbol{P}_1 \cdot \nabla \boldsymbol{B} = \tfrac{1}{2} \boldsymbol{b} \wedge \nabla \boldsymbol{B}.$$

The second average is a little bit more complicated. We first note

$$\begin{aligned} \boldsymbol{n}_1 \wedge (\boldsymbol{n}_1 \cdot \nabla)\boldsymbol{b} &= \boldsymbol{n}_2 \left[\boldsymbol{n}_2 \cdot \boldsymbol{n}_1 \wedge (\boldsymbol{n}_1 \cdot \nabla)\boldsymbol{b} \right] + \boldsymbol{b} \left[\boldsymbol{b} \cdot \boldsymbol{n}_1 \wedge (\boldsymbol{n}_1 \cdot \nabla)\boldsymbol{b} \right] \\ &= \boldsymbol{n}_2 (\boldsymbol{n}_1 \cdot \nabla)\boldsymbol{b} \cdot (\boldsymbol{n}_2 \wedge \boldsymbol{n}_1) + \boldsymbol{b} (\boldsymbol{n}_1 \cdot \nabla)\boldsymbol{b} \cdot (\boldsymbol{b} \wedge \boldsymbol{n}_1) \\ &= -\boldsymbol{n}_2 \boldsymbol{n}_1 \cdot \nabla \boldsymbol{b} \cdot \boldsymbol{b} + \boldsymbol{b} \boldsymbol{n}_1 \cdot \nabla \boldsymbol{b} \cdot \boldsymbol{n}_2 = \boldsymbol{b}(\boldsymbol{n}_2 \boldsymbol{n}_1) : (\nabla \boldsymbol{b}), \end{aligned}$$

* It may help to note that this statement is not true for the combination $\boldsymbol{n}_1 \cdot \nabla \boldsymbol{n}_2$!

where we used (6.19) and (4.10). Using now (6.25) and (6.29) we get

$$\overline{\boldsymbol{n}_1 \wedge (\boldsymbol{n}_1 \cdot \nabla)\boldsymbol{b}} = \tfrac{1}{2}\boldsymbol{b}\, \boldsymbol{P}_1 : \nabla \boldsymbol{b} = \tfrac{1}{2}\boldsymbol{b}\, \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}).$$

Collecting these results, we find

$$\dot{\boldsymbol{Y}} = \left(v_{\parallel} + \frac{\epsilon}{2\Omega} v_{\perp}^2 \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}) \right) \boldsymbol{b}$$

$$+ \epsilon c \frac{\boldsymbol{E} \wedge \boldsymbol{B}}{B^2} + \epsilon \frac{v_{\perp}^2}{2\Omega B} \boldsymbol{b} \wedge \nabla B + \epsilon \frac{v_{\parallel}^2}{\Omega} \boldsymbol{b} \wedge (\boldsymbol{b} \cdot \nabla) \boldsymbol{b}. \tag{6.37}$$

Let us discuss this important equation, which provides us with the expression of the *average drift velocity of the guiding centre*, through order ϵ. The right-hand side contains a first term, representing the parallel motion (it will be discussed afterwards) and three terms which are all perpendicular to $\boldsymbol{b}$. The first of these terms represents the familiar *electric drift*, which was found in (5.22) for the uniform case. The second term represents the *grad-B drift*, and was also found from a qualitative argument in (5.24): we now possess its exact expression. The third term is a new effect, called *centrifugal drift*. In order to understand its meaning (Sivukhin 1965), consider a magnetic field line and its local Frenet triad at point $\boldsymbol{x}$ (fig. 6.1). In a curved field, the particle tends to follow the field line (because its velocity has a component $v_{\parallel}$). As a result, its motion has a component which is a *rotation* around the instantaneous centre of curvature C. The corresponding angular velocity vector $\boldsymbol{\omega}$ is such that

$$\boldsymbol{v}_{\parallel} = \boldsymbol{\omega} \wedge \boldsymbol{\rho} = -\rho \boldsymbol{\omega} \wedge \boldsymbol{N}, \tag{6.38}$$

where $\boldsymbol{N}$ is the unit vector along the principal normal and ρ is the curvature

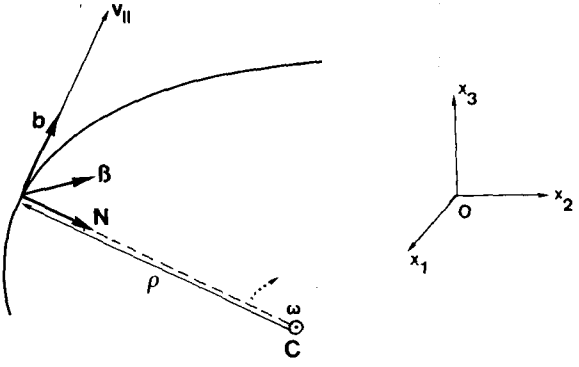

Fig. 6.1. Origin of the centrifugal drift.

radius. It follows that the angular velocity is directed along the binormal (3.19):

$$\boldsymbol{\omega} = \frac{v_\parallel}{\rho}(\boldsymbol{b} \wedge \boldsymbol{N}) = \frac{v_\parallel}{\rho}\boldsymbol{\beta}. \tag{6.39}$$

The instantaneous motion of the particle may be referred to a reference frame centered at the centre of curvature C. The relation between the velocity $\boldsymbol{v}$ referred to the absolute frame and the velocity $\boldsymbol{v}_{\text{rel}}$ relative to the rotating frame is

$$\boldsymbol{v} = \boldsymbol{v}_{\text{rel}} + \boldsymbol{\omega} \wedge \boldsymbol{\rho}. \tag{6.40}$$

Hence, from (6.38),

$$\boldsymbol{v}_{\text{rel}} = \boldsymbol{v}_\perp . \tag{6.41}$$

In this rotating frame the magnetic field is straight, but the frame is clearly not inertial. Therefore, in writing the corresponding equations of motion, we must take account of the well-known fictitious forces (see e.g. Landau and Lifshitz 1957),

$$m\dot{\boldsymbol{v}}_\perp = \boldsymbol{f} + 2m(\boldsymbol{v}_\perp \wedge \boldsymbol{\omega}) - m\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\rho}) - m\dot{\boldsymbol{\omega}} \wedge \boldsymbol{\rho}. \tag{6.42}$$

The last term, involving $\dot{\boldsymbol{\omega}}$, can be shown to be of higher order. The Coriolis force is easily evaluated:

$$\begin{aligned} \boldsymbol{F}_{\text{cor}} &= 2m\boldsymbol{v}_\perp \wedge \boldsymbol{\omega} = 2m\left(v_{\text{N}}\boldsymbol{N} + v_\beta\boldsymbol{\beta}\right) \wedge \boldsymbol{\omega} \\ &= 2mv_{\text{N}}\boldsymbol{N} \wedge \boldsymbol{\omega} = 2mv_{\text{N}}\frac{v_\parallel}{\rho}(\boldsymbol{N} \wedge \boldsymbol{\beta}) = 2m\frac{v_{\text{N}}v_\parallel}{\rho}\boldsymbol{b}. \end{aligned} \tag{6.43}$$

Thus, the Coriolis force is parallel to the magnetic field, and therefore produces no perpendicular drift. The centrifugal force is evaluated using (6.39):

$$\boldsymbol{F}_{\text{cent}} = -m\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\rho}) = m\omega^2\boldsymbol{\rho} = -m\frac{v_\parallel^2}{\rho}\boldsymbol{N}. \tag{6.44}$$

This force is directed along the principal normal. It produces a drift which can be evaluated from formula (5.22) by replacing the electric force $e\boldsymbol{E}$ by the centrifugal force (6.44)

$$\boldsymbol{w}_{\text{cent}} = \frac{c}{eB^2}(\boldsymbol{F}_{\text{cent}} \wedge \boldsymbol{B}) = -\frac{mc}{eB}\frac{v_\parallel^2}{\rho}\boldsymbol{N} \wedge \boldsymbol{b} = \frac{1}{\Omega}\frac{v_\parallel^2}{\rho}\boldsymbol{\beta}. \tag{6.45}$$

Recalling the geometrical relations (3.18) and (3.19), we see that this drift velocity, directed along the binormal of the field line, agrees precisely with the last term of (6.37).

The interpretation of the "parallel drift velocity" leads to some serious difficulties. The naive interpretation would tell us that the parallel velocity of the guiding centre equals the parallel velocity of the particle. The occurrence of a correction of order ϵ is then surprising. Sivukhin (1965) interprets it by trying to show that the *average* parallel guiding centre velocity equals the *average* velocity of the particle, but his proof is not convincing *. It is very striking to note that this "parallel drift" term has received different forms (and values!) according to the various authors (see e.g. the lucid discussion by Wimmel 1982).

When we address the higher order terms in the method of the average, and in particular the averaged equations of motion for $v_{\parallel}$, $v_{\perp}$, φ, the confusion in the literature appears even more appalling. We shall therefore introduce in the next sections a different, elegant and reliable method for treating the problem.

1.7. *The drift approximation: The averaging pseudo-canonical transformation. I. Stationary, homogeneous fields*

Besides its technical difficulties and ambiguities, the method of the average has a more fundamental disadvantage. In this method, the main tool for suppressing the "odious oscillations" is the *gyrophase average* (6.12) (or, even more precisely, the average over rapid oscillations in time). This concept has been introduced in the study of a differential system (6.3)–(6.4) containing a small parameter. There is no objection to its use in finding asymptotic solutions of such equations. However, in the problems to be studied in this book, we must be more ambitious. We are not only – and even not primarily – interested in finding detailed trajectories of a single charged particle. To us this problem is just an introduction to the much more general problem of setting up a description of a many-particle system – a plasma – with interactions. For this purpose, we do not need a method of solution of a set of differential equations; we rather need a deep insight into the *structure* of these equations. A crucial fact in this respect is the *Hamiltonian structure* of the starting equations (table 6.1). This property is exceedingly important for our

* He proves that $\overline{\boldsymbol{v}\cdot\boldsymbol{b}(\boldsymbol{Y})} = \overline{(\mathrm{d}\boldsymbol{Y}/\mathrm{d}t)\cdot\boldsymbol{b}(\boldsymbol{q})}$, whereas the statement should be expressed as $\overline{\boldsymbol{v}\cdot\boldsymbol{b}(\boldsymbol{q})} = \overline{(\mathrm{d}\,\boldsymbol{Y}/\mathrm{d}t)\cdot\boldsymbol{b}(\boldsymbol{Y})}$, which is not true.

final goal, because it ensures the validity of Liouville's theorem. It thus allows a correct transition from an individual-trajectory description to the phase-space distribution function description, which opens the gate to statistical mechanics, and thus to plasma physics.

It is clear, on the other hand, that the gyrophase average (6.12), and, a fortiori, the time-smoothing concepts, *do not seem to fit into the structure of Hamiltonian dynamics*, as exposed in section 1.2. A relation such as (6.18) between the new variable Y and the old variable y involves, in principle, the following steps:

(a) solve the differential system in order to obtain $y(t)$;

(b) separate in the solution two time scales, one fast, t_f, and one slow, t_s: $y(t) = y(t_f, t_s)$,

(c) average by integration over the fast time: $Y(t_s) = y(t_f, t_s)$. This procedure is markedly different from the canonical or pseudo-canonical transformations of the Hamiltonian formalism (2.24): these are functional relations between the old and new variables, evaluated at the *same* time. In other words, the pseudo-canonical transformations are point-transformations in phase space, while the average relates the new variables to integrals over a trajectory, i.e. a curve in phase space.

In conclusion, the application of the method of the average to the equations of motion (5)–(8) in table 6.1 destroys their Hamiltonian structure. As a result, an enormous amount of work done on the statistical physics and transport theory of plasmas in an external magnetic field, in which the drift approximation plays a crucial role, appeared to have been constructed on a loose basis. Indeed, the very first fundament: the Liouville equation, could not be proved, and appeared as a kind of ad hoc construction. This is not a purely academic problem. A non-Hamiltonian system may have very peculiar properties: attractors, instabilities and even strange attractors. When one realizes the importance of such phenomena in plasma physics, and the delicacy required in their study, one must be sure at least that such properties have not been introduced artificially *.

The history of the connection between the drift approximation and the Hamiltonian dynamics is interesting. It is important to note that the first paper dealing in *full generality* with the suppression of the fast oscillations in the class of differential equations describing a nearly periodic motion is due to Kruskal (1962). The recursive algorithm proposed in that paper for construct-

* This discussion should not be interpreted as our dismissing altogether all averaging methods. We will see later in this book that averaging is an important tool in many fields of plasma physics, and, in particular, in transport theory. But averages should be taken at the proper time and on a physically sound basis. The point is that at this stage of the theory, when the very starting point of the physical description is being constructed, one should avoid performing uncontrollable operations.

ing the averaging transformation is very close to the method of the average. In his work, Kruskal proved a number of important properties of this averaging transformation, one of which is of basic interest in the present context. He proved that *the Hamiltonian form is a hereditary property*. More explicitly, if the initial variables (x_k, φ) can be grouped in *canonical* pairs $(\boldsymbol{q}, \boldsymbol{p})$ and if the differential system (6.3), (6.4) has the Hamiltonian form (2.2), the averaging transformation can be constructed as a *canonical transformation* to a set of variables $(\boldsymbol{Q}^c, \boldsymbol{P}^c)$ (one of which is the gyrophase), such that their equations of motion also have the Hamiltonian form. In his section devoted to "Poisson brackets", Kruskal explicitly states the conditions (7.1) which will be used below; thus Kruskal's philosophy is very close to the idea of pseudo-canonical transformations. However, his results are of the type of "existence theorems", which were not worked out explicitly. The application of his results in the paper by Northrop et al. (1966) did not make use of Hamiltonian methods.

At that time, a large amount of work was devoted to a parallel line of research, which is related to the present problem. This is the search for *adiabatic invariants of motion* (which will be discussed in section 1.8) [see, e.g. Alfven (1950), Chandrasekhar (1958), Gardner (1959), Bernstein (1971); a good account of the early history of the problem is given by Kruskal (1962)]. In this problem, an extensive use was made of Hamiltonian methods. But as only "orthodox" canonical transformations were allowed, the authors were led to use global geometrical characteristics of the field lines as canonical variables, which led to a rather impractical formalism.

The interest in these problems faded more or less away in the 1970's, until a breakthrough happened with the publication of Littlejohn's (1979, 1981, 1983) papers. In these papers, Kruskal's statement was fully confirmed *: *The drift approximation fits perfectly in Hamiltonian dynamics*. And, most important, the theory was made fully operational.

Littlejohn's results, while coinciding with the dominant order results of the average method, went beyond it and yielded, for the first time, a complete and coherent description of the dynamics of a charged particle in the drift approximation. These works have given a new impetus to the research in this field.

One may wonder why such an important result had to await so long before discovery. The reason, in our view, is clear. As long as Hamiltonian dynamics was dominated by the "dogma" of canonical transformations, one could not even think of approaching this problem: we have seen that even the use of the physical velocity of a particle requires a non-canonical transformation. Only when this psychological (rather than mathematical) barrier was lifted and one

* Strangely, Kruskal's (1962) paper is quoted only in passing by Littlejohn (1981), and not in connection with his fundamental theorem on the hereditary property of Hamiltonian structure.

started using pseudo-canonical transformations in Hamiltonian dynamics, did one gain enough flexibility for treating, in particular, the problem of the guiding centre motion.

The fundamental result of Hamiltonian guiding centre motion theory may be called:

*The **Kruskal–Littlejohn theorem**.*

Let $z^k (k = 1, \ldots, 6)$ be the set of particle phase-space coordinates

$$z^k \equiv \{\boldsymbol{q},\ v_{\parallel},\ v_{\perp},\ \varphi\}.$$

There exists a pseudo-canonical transformation to a new set of phase-space coordinates $Z^k \equiv \{\boldsymbol{Y},\ U,\ W,\ \phi\}$,

$$Z^k = Z^k(z^1, \ldots, z^6), \tag{7.1}$$

such that

– the Lie brackets of the new phase-space coordinates are all independent of the new gyrophase $\phi \equiv Z^6$:

$$[Z^k,\ Z^m] = \Sigma^{km}(Z^1, \ldots, Z^5); \tag{7.2}$$

– the transformed Hamiltonian is independent of the new gyrophase $\phi \equiv Z^6$,

$$H = H(Z^1, \ldots, Z^5). \tag{7.3}$$

If this theorem holds, an immediate corollary is the following:

Whatever the form of the Hamiltonian, expressed in terms of the new variables, the equations of motion are

$$\dot{Z}^k = [Z^k,\ H] = F^k(Z^1, \ldots, Z^5), \tag{7.4}$$

where the functions F^k are independent of the new gyrophase $\phi \equiv Z^6$.

The goal formulated at the beginning of section 1.6, i.e. the elimination of the "odious dependence" on the gyrophase is thus realized by a pseudo-canonical transformation which preserves the Hamiltonian structure of the theory. No use of a gyrophase-average (or, a fortiori, a time-average) operation is necessary for achieving this purpose. The transformation $\{\boldsymbol{q},\ v_{\parallel},\ v_{\perp},\ \varphi) \rightarrow \{\boldsymbol{Y},\ U,\ W,\ \phi\}$ will be called an *averaging pseudo-canonical transformation*; the new variables will be called *averaging* (not "averaged"!) *variables*.

The averaging pseudo-canonical transformation is *not unique*: there exist infinitely many sets of variables possessing the required properties. Indeed,

Let Z^k $(k=1,\ldots,6)$ *be a known set of averaging variables, and consider the following six functions of* Z^k:

$$\hat{Z}^\lambda = \hat{Z}^\lambda(Z^1,\ldots,Z^5),$$

$$\hat{Z}^6 = Z^6 + \zeta(Z^1,\ldots,Z^5), \quad \lambda = 1,\ldots,5, \tag{7.5}$$

(where we recall that $Z^6 \equiv \phi$). *It is easily checked that the variables* $\hat{Z}^1,\ldots,\hat{Z}^6$ *also satisfy the requirements of the Kruskal–Littlejohn theorem*:

$$[\hat{Z}^k, \hat{Z}^m] = \hat{\Sigma}^{km}(\hat{Z}^1,\ldots,\hat{Z}^5), \quad k, m = 1,\ldots,6. \tag{7.6}$$

Kruskal (1962) proved that conditions (7.5) are necessary and sufficient.

We now show, for a particularly simple example, how an averaging pseudo-canonical transformation can be constructed. Consider a charged particle in the presence of a *homogeneous* magnetic field $\boldsymbol{B}$ and a *homogeneous* electric field $\boldsymbol{E}$. In this case, the fundamental Lie brackets (1) of table 6.1 reduce to

$$[q_i, q_j] = 0 \qquad [\boldsymbol{q}, v_\parallel] = m^{-1}\boldsymbol{b}, \quad [\boldsymbol{q}, v_\perp] = m^{-1}\boldsymbol{n}_1(\varphi),$$
$$[\boldsymbol{q}, \varphi] = -(mv_\perp)^{-1}\boldsymbol{n}_2(\varphi), \quad [v_\parallel, v_\perp] = 0, \qquad [v_\parallel, \varphi] = 0,$$
$$[v_\perp, \varphi] = -\frac{1}{\epsilon}\frac{\Omega}{mv_\perp}. \tag{7.7}$$

All the right-hand sides are independent of $\boldsymbol{q}$ in the present case, but two of them are oscillating functions of φ. The Hamiltonian is

$$H(\boldsymbol{q}, v_\parallel, v_\perp) = \tfrac{1}{2}m\left(v_\parallel^2 + v_\perp^2\right) - e\boldsymbol{E}\cdot\boldsymbol{q}. \tag{7.8}$$

We now seek a *pseudo-canonical transformation*, defined as a *power series in the small parameter* ϵ, and limit ourselves to the first order expansions

$$\boldsymbol{Y} = \boldsymbol{q} + \epsilon\boldsymbol{\rho}(\boldsymbol{q}, v_\parallel, v_\perp, \varphi) + O(\epsilon^2),$$
$$U = v_\parallel + \epsilon v_\parallel(\boldsymbol{q}, v_\parallel, v_\perp, \varphi) + O(\epsilon^2),$$
$$W = v_\perp + \epsilon v_\perp(\boldsymbol{q}, v_\parallel, v_\perp, \varphi) + O(\epsilon^2),$$
$$\phi = \varphi + \epsilon\psi(\boldsymbol{q}, v_\parallel, v_\perp, \varphi) + O(\epsilon^2). \tag{7.9}$$

We want to determine the functions ρ, $v_{\|}$, $v_{\perp}$, ψ in such a way that the *Lie brackets of the new variables be independent of* ϕ. As we know that the averaging transformation, if it exists, is not unique, it is sufficient to determine a *particular* set of such functions; it is not necessary to solve the forthcoming equations in full generality (including all the integration constants). If one particular set of averaging variables is found, eqs. (7.5) enable us to construct *any* other set of such variables. In particular, we shall try to impose an additional, simplifying and aesthetically satisfactory condition: we want the *Hamiltonian to be form-invariant* through order ϵ:

$$H(\boldsymbol{Y}, U, W) = \tfrac{1}{2}m(U^2 + W^2) - e\boldsymbol{E}\cdot\boldsymbol{Y} + \mathrm{O}(\epsilon^2). \tag{7.10}$$

From (7.10) and (7.9) we find that the condition of form-invariance requires

$$mv_{\|}\nu_{\|} + mv_{\perp}\nu_{\perp} - e\boldsymbol{\rho}\cdot\boldsymbol{E} = 0,$$

which is satisfied by taking

$$\begin{aligned} \nu_{\|} &= v_{\perp}\beta(\boldsymbol{q}, v_{\|}, v_{\perp}, \varphi), \\ \nu_{\perp} &= -v_{\|}\beta(\boldsymbol{q}, v_{\|}, v_{\perp}, \varphi) + \frac{e}{mv_{\perp}}\boldsymbol{\rho}\cdot\boldsymbol{E}, \end{aligned} \tag{7.11}$$

introducing the new unknown function β. We now consider the seven Lie brackets in turn, starting with those containing oscillating functions in the old variables. Thus

$$\begin{aligned} [\boldsymbol{Y}, W] &= \left[\boldsymbol{q} + \epsilon\boldsymbol{\rho},\ v_{\perp} - \epsilon v_{\|}\beta + \epsilon\frac{e}{mv_{\perp}}\boldsymbol{\rho}\cdot\boldsymbol{E}\right] \\ &= [\boldsymbol{q}, v_{\perp}] + \epsilon\left([\boldsymbol{\rho}, v_{\perp}] - [\boldsymbol{q}, v_{\|}\beta] + \frac{e}{m}[\boldsymbol{q}, v_{\perp}^{-1}\boldsymbol{\rho}\cdot\boldsymbol{E}]\right) \\ &\quad + \epsilon^2\left(-[\boldsymbol{\rho}, v_{\|}\beta] + \frac{e}{m}[\boldsymbol{\rho}, v_{\perp}^{-1}\boldsymbol{\rho}\cdot\boldsymbol{E}]\right). \end{aligned}$$

We now order these terms in powers of ϵ (remember that $[v_{\perp}, \varphi] = \mathrm{O}(\epsilon^{-1})$!):

$$[\boldsymbol{Y}, W] = [\boldsymbol{q}, v_{\perp}] + \epsilon\frac{\partial\boldsymbol{\rho}}{\partial\varphi}[\varphi, v_{\perp}] + \mathrm{O}(\epsilon) = \frac{1}{m}\boldsymbol{n}_1 + \frac{\Omega}{mv_{\perp}}\frac{\partial\boldsymbol{\rho}}{\partial\varphi} + \mathrm{O}(\epsilon).$$

To order ϵ^0, the oscillations of $\boldsymbol{n}_1(\varphi)$ are compensated if we take

$$\frac{\partial\boldsymbol{\rho}}{\partial\varphi} = -\frac{v_{\perp}}{\Omega}\boldsymbol{n}_1$$

to which could be added a vector independent of φ (but we do not need the general solution!). From (4.12) we immediately find

$$\rho = -\frac{v_\perp}{\Omega}\boldsymbol{n}_2. \tag{7.12}$$

Before checking the absence of oscillations to order ϵ, we evaluate other Lie brackets. The calculation of $[Y, \phi]$ does not teach us anything new: we find that the choice (7.12) also suppresses the oscillations of this bracket to order ϵ^0. We must now check that the correction ρ does not introduce any oscillations into brackets which were not oscillating in the old variables. In particular

$$[W, \phi] = [v_\perp, \varphi] + \epsilon[v_\perp, \varphi]\left(-v_\parallel\frac{\partial\beta}{\partial v_\perp} + \frac{e}{m}\boldsymbol{E}\cdot\frac{\partial(\rho/v_\perp)}{\partial v_\perp} + \frac{\partial\psi}{\partial\varphi}\right)$$

$$= -\frac{1}{\epsilon}\frac{\Omega}{mv_\perp} + \frac{\Omega}{mv_\perp}\left(v_\parallel\frac{\partial\beta}{\partial v_\perp} - \frac{\partial\psi}{\partial\varphi}\right).$$

Here we must be careful: we require the suppression of the oscillations in the Lie bracket expressed in terms of the *new* variables. As a result, we must express $v_\perp$ in terms of W in the dominant (ϵ^{-1}) term. This results in additional, oscillating corrections of order ϵ^0,

$$[W, \phi] = -\frac{1}{\epsilon}\frac{\Omega}{mW\left(1 + \epsilon\frac{v_\parallel}{v_\perp}\beta + \epsilon\frac{e}{m\Omega v_\perp}\boldsymbol{n}_2\cdot\boldsymbol{E}\right)} + \frac{\Omega}{mv_\perp}(\quad)$$

$$= -\frac{1}{\epsilon}\frac{\Omega}{mW} + \frac{\Omega}{mv_\perp}\left(\frac{v_\parallel}{v_\perp}\beta + v_\parallel\frac{\partial\beta}{\partial v_\perp} + \frac{e}{m\Omega v_\perp}\boldsymbol{E}\cdot\boldsymbol{n}_2(\varphi) - \frac{\partial\psi}{\partial\rho}\right)$$

$$+ O(\epsilon).$$

The oscillations introduced by $\boldsymbol{n}_2$ in order ϵ^0 are compensated by the following simple choice

$$\beta = 0, \qquad \psi = -\frac{e}{m\Omega v_\perp}\boldsymbol{n}_1\cdot\boldsymbol{E}. \tag{7.13}$$

Equations (7.11)–(7.13) provide a complete solution to the problem; but we must check that this *particular* solution is sufficient for suppressing the oscillations in all the fundamental Lie brackets, not only to order ϵ^0, but also

to order ϵ. A direct calculation of these brackets shows that this is indeed the case. We thus collect the results

$$\begin{aligned} \boldsymbol{Y} &= \boldsymbol{q} - \epsilon \frac{v_\perp}{\Omega} \boldsymbol{n}_2(\varphi) + \mathrm{O}(\epsilon^2), \\ U &= v_\parallel + \mathrm{O}(\epsilon^2), \\ W &= v_\perp - \epsilon \frac{e}{m\Omega} \boldsymbol{E} \cdot \boldsymbol{n}_2(\varphi) + \mathrm{O}(\epsilon^2), \\ \phi &= \varphi - \epsilon \frac{e}{m\Omega v_\perp} \boldsymbol{E} \cdot \boldsymbol{n}_1(\varphi) + \mathrm{O}(\epsilon^2). \end{aligned} \tag{7.14}$$

Before continuing, we recognize that the new variable $\boldsymbol{Y}$ is nothing other than the *coordinate of the guiding centre* (6.16), which entered the problem quite naturally.

The fundamental Lie brackets are

$$\begin{aligned} &[Y_i, Y_j] = -\epsilon \frac{c}{eB} \varepsilon_{ijk} b_k, \qquad [\boldsymbol{Y}, U] = \frac{1}{m} \boldsymbol{b}, \qquad [\boldsymbol{Y}, W] = 0, \\ &[\boldsymbol{Y}, \phi] = 0, \qquad [U, W] = 0, \qquad [U, \phi] = 0, \\ &[W, \phi] = -\frac{1}{\epsilon} \frac{\Omega}{mW}, \end{aligned} \tag{7.15}$$

where ε_{ijk} is, as usual, the completely antisymmetric Levi–Civita symbol. These brackets are correct through order ϵ (ϵ^0 for [W, ϕ]). We clearly see that they do not contain any oscillating function, hence (7.14) is, indeed, *an averaging pseudo-canonical transformation*.

We finally derive the equations of motion from the Hamiltonian (7.10), using the new brackets (7.15). For instance

$$\begin{aligned} \dot{Y}_i &= [Y_i, H] = m\big(U[Y_i, U] + W[Y_i, W]\big) - e[Y_i, Y_j] E_j \\ &= U b_j + 0 + \epsilon \frac{c}{B} \varepsilon_{ijk} E_j b_k, \end{aligned}$$

which can be rewritten as a vector equation,

$$\dot{\boldsymbol{Y}} = U\boldsymbol{b} + \epsilon c \frac{\boldsymbol{E} \wedge \boldsymbol{B}}{B^2}. \tag{7.16}$$

We similarly obtain

$$\dot{U} = \frac{e}{m} \boldsymbol{E} \cdot \boldsymbol{B}, \tag{7.17}$$

$$\dot{W} = 0, \tag{7.18}$$

$$\dot{\phi} = \frac{1}{\epsilon} \Omega. \tag{7.19}$$

These are the equations of motion in the drift approximation through order ϵ (ϵ^0 for $\dot{\phi}$), for homogeneous fields. Equation (7.16) describes the guiding centre velocity as a superposition of a parallel velocity U (equal to the particle velocity to this order) and a perpendicular drift velocity of order ϵ. The latter is the familiar electric drift velocity found previously (5.22). The guiding centre is uniformly accelerated by the parallel component of the electric field. Finally, the new gyrophase still oscillates at the constant Larmor frequency Ω. We have thus obtained systematically all the previous results. We also note that eq. (7.16) coincides exactly with eq. (6.37) for a homogeneous field. This demonstrates that the method of the average and the pseudo-canonical averaging transformation yield identical results in this case. We note, however, that the explicit relations (7.14), as well as the additional equations of motion (7.17)–(7.19) come out very simply in the latter method.

1.8. The drift approximation: The averaging pseudo-canonical transformation. II. Stationary, spatially inhomogeneous fields

Having illustrated in detail the Kruskal–Littlejohn theorem, i.e. the existence of an averaging pseudo-canonical transformation, for the simple case of homogeneous magnetic and electric fields, we go over to the case of arbitrarily inhomogeneous, but still stationary, fields.

In this case, the magnetic field can be written in the form

$$\boldsymbol{B}(\boldsymbol{x}) = B(\boldsymbol{x})\, \boldsymbol{b}(\boldsymbol{x}). \tag{8.1}$$

Hence, the intensity $B(\boldsymbol{x})$ is spatially inhomogeneous, and the orientation of the unit vector $\boldsymbol{b}(\boldsymbol{x})$ is also space-dependent. This means that *the magnetic field lines are curved and twisted*. In spite of the additional mathematical complication, it is essential to devise methods for treating this case. It is well known that a straight magnetic field cannot confine a plasma. Hence all thermonuclear fusion devices (in particular, the toroidal ones, such as the tokamak) are based on cleverly designed, curved and twisted magnetic field configurations. In the other important domain of application of plasma

physics, it is clear to anyone who has looked at a picture of a solar flare, or of the solar corona, or of the earth's magnetosphere, that curvature and torsion are universal features of magnetic fields in nature.

In order to treat this problem, Littlejohn, in his second paper (1981), uses a "geometrical" method, based on an extension of a classical method of Hamiltonian dynamics: the *Darboux transformation*. Littlejohn's method is not simple, either from the basic point of view or for the calculations. In his third paper Littlejohn (1983) introduced a different approach for treating the problem. This new, "variational" method is of a Lagrangian, rather than a Hamiltonian nature (although the connection with Hamiltonian dynamics is made explicit at the end). This treatment is, pragmatically speaking, simpler, but also mathematically less elegant, than the 1981 paper.

Recently, Weyssow and Balescu (1986) showed that the explicit construction of the Kruskal–Littlejohn averaging pseudocanonical transformation can be done in a very direct and conceptually simple way. The idea is the extension of the method illustrated in section 1.7 for the homogeneous case. The new variables are written in the form of series expansions, like (7.9), and the unknown functions are determined by satisfying conditions (7.2) to each order. The terms of the transformation were determined through second order in ϵ (third order for $\boldsymbol{Y}$), for general magnetic and electric fields, which may also depend slowly on the time (see section 1.9).

It is impossible to expose here the detail of the calculations in the amount of space available. We thus choose to expose and comment the main results, through order ϵ; these can be checked in various ways *.

The basic *averaging pseudo-canonical transformation* is defined by the following equations:

$$\boldsymbol{Y} = \boldsymbol{q} - \frac{\epsilon}{\Omega} v_\perp \boldsymbol{n}_2 + \left(\frac{\epsilon}{\Omega}\right)^2 v_\perp^2 \times \left(\tfrac{3}{8} \boldsymbol{b} \left[\boldsymbol{n}_2 \cdot (\nabla \wedge \boldsymbol{n}_1) + \boldsymbol{n}_1 \cdot (\nabla \wedge \boldsymbol{n}_2) \right] + \frac{v_\parallel}{v_\perp} (\boldsymbol{n}_2 \boldsymbol{b} + 2 \boldsymbol{b} \boldsymbol{n}_2) \cdot (\nabla \wedge \boldsymbol{b}) - \frac{1}{4B} (\boldsymbol{n}_2 \boldsymbol{n}_2 - \boldsymbol{n}_1 \boldsymbol{n}_1) \cdot \nabla B \right), \quad (8.2)$$

* We have already stressed the non-unicity of the averaging transformation. It turns out that in the successive papers by Littlejohn, different versions of the averaging variables are given. Unfortunately, there are also some misprints and some small internal inconsistencies in his papers (especially in the one from 1983). Also, his use of artificial units with $e = m = c = 1$ is a little bit irritating. The set of variables described here under the name of "natural guiding centre variables" (to be distinguished from Littlejohn's (1983) "standard guiding centre variables") are close, but not identical to those in his paper of 1981.

$$U = v_\parallel + \frac{\epsilon}{\Omega} v_\perp^2 \left(\tfrac{1}{4} [\boldsymbol{n}_1 \cdot (\nabla \wedge \boldsymbol{n}_1) - \boldsymbol{n}_2 \cdot (\nabla \wedge \boldsymbol{n}_2) + 2\boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})] + \frac{v_\parallel}{v_\perp} \boldsymbol{n}_1 \cdot (\nabla \wedge \boldsymbol{b}) \right), \quad (8.3)$$

$$W = v_\perp + \frac{\epsilon}{\Omega} v_\parallel v_\perp \left(-\tfrac{1}{4} [\boldsymbol{n}_1 \cdot (\nabla \wedge \boldsymbol{n}_1) - \boldsymbol{n}_2 \cdot (\nabla \wedge \boldsymbol{n}_2) + 2\boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})] - \frac{v_\parallel}{v_\perp} \boldsymbol{n}_1 \cdot (\nabla \wedge \boldsymbol{b}) + \frac{v_\perp}{v_\parallel} \frac{e}{m} \boldsymbol{n}_2 \cdot \nabla \Phi \right), \quad (8.4)$$

$$\phi = \varphi + \frac{e}{\Omega} \left(\frac{v_\parallel^2}{v_\perp} \boldsymbol{n}_2 \cdot (\nabla \wedge \boldsymbol{b}) - v_\perp \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{n}_2) + \tfrac{1}{4} v_\parallel [\boldsymbol{n}_2 \cdot (\nabla \wedge \boldsymbol{n}_1) + \boldsymbol{n}_1 \cdot (\nabla \wedge \boldsymbol{n}_2)] + v_\perp B^{-1} \boldsymbol{n}_1 \cdot \nabla B + \frac{e}{m v_\perp} \boldsymbol{n}_1 \cdot \nabla \Phi \right). \quad (8.5)$$

Here all the fields (Ω, B, Φ, $\boldsymbol{b}$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$) are evaluated at the point $\boldsymbol{x} = \boldsymbol{q}$ and at phase φ, and $\nabla \equiv \partial / \partial \boldsymbol{q}$. These equations reduce to (7.14) for homogeneous fields. The usual characteristics of the pseudo-canonical averaging transformation are collected in table 8.1.

The fundamental Lie brackets (1) in table 8.1 are clearly free from oscillating terms. Indeed, the oscillating functions $\boldsymbol{n}_i(\boldsymbol{Y}, \phi)$ enter these expressions only through the combination $\boldsymbol{R} \equiv \nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_2$ which, as we know from (6.32), is independent of the gyrophase ϕ. Thus, (8.2–8.5) define indeed an averaging pseudo-canonical transformation. We urge the reader to check some of the fundamental Lie brackets of table 8.1; it is a non-trivial exercise in the manipulation of the local basis vectors. He will see how exquisitely delicate are the compensations of the oscillating terms, leading to eqs. (1) *.

* The following remark is interesting. The results (8.2)–(8.5), given here through order ϵ, are insufficient for the complete calculation of some of the brackets through order ϵ (e.g. $[U, W]$, $[W, \phi]$, $[U, \phi]$). Indeed, as pointed out in section 1.7, the ϵ-ordering of the *brackets* is not the same as the ϵ-ordering of the *functions* on which the bracket operates (this is due to the existence of the "large" elementary bracket $[v_\perp, \varphi] = O(\epsilon^{-1})$). As a result, the calculation of some brackets through order ϵ requires the knowledge of the dynamical variables Z^k through order ϵ^2. The ϵ^2-contributions were determined explicitly by Weyssow and Balescu (1986).

Table 8.1
Natural guiding centre variables $\boldsymbol{Y}, U, W, \phi$.

Definition See eqs. (8.2)–(8.5)

Fundamental Lie brackets

$$[Y_i, Y_j] = -\epsilon \frac{1}{m\Omega} \varepsilon_{ijk} b_k, \qquad [\boldsymbol{Y}, U] = m^{-1} \boldsymbol{b}^*, \qquad [\boldsymbol{Y}, W] = \epsilon \frac{W}{2mB\Omega} \boldsymbol{b} \wedge \nabla B,$$

$$[\boldsymbol{Y}, \phi] = \epsilon \frac{1}{m\Omega} \boldsymbol{b} \wedge \boldsymbol{R}, \qquad [U, W] = -\frac{W}{2mB} \boldsymbol{b}^* \cdot \nabla B,$$

$$[U, \phi] = -\frac{1}{m} \boldsymbol{b}^* \cdot \boldsymbol{R} + \frac{1}{2m} \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}),$$

$$[W, \phi] = -\frac{1}{\epsilon} \frac{\Omega}{mW} + \epsilon \frac{W}{2mB\Omega} (\nabla B) \cdot (\nabla \wedge \boldsymbol{b}). \tag{1}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{v}) \to (\boldsymbol{Y}, U, W, \phi)$

$$|J| = \left(B_{\|}^*/B\right) W. \tag{2}$$

Hamiltonian

$$H = (m/2)(U^2 + W^2) + e\Phi(\boldsymbol{Y}). \tag{3}$$

Equations of motion

$$\dot{\boldsymbol{Y}} = U \boldsymbol{b}^*, \tag{4}$$

$$\dot{U} = -\frac{W^2}{2B} \boldsymbol{b}^* \cdot \nabla B - \frac{e}{m} \boldsymbol{b}^* \cdot \nabla \Phi, \tag{5}$$

$$\dot{W} = \frac{UW}{2B} \boldsymbol{b}^* \cdot \nabla B - \epsilon \frac{cW}{2B^2} (\nabla \Phi) \cdot (\boldsymbol{b} \wedge \nabla B), \tag{6}$$

$$\dot{\phi} = \epsilon^{-1} \Omega + U \boldsymbol{b} \cdot \boldsymbol{R} - \tfrac{1}{2} U \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}). \tag{7}$$

Remarks

All fields $\Omega, B, \Phi, \boldsymbol{b}, \boldsymbol{b}^*, \boldsymbol{b}^{**}$ are evaluated at $\boldsymbol{x} = \boldsymbol{Y}$; $\nabla \equiv \partial/\partial \boldsymbol{Y}$.

$$\boldsymbol{b}^* = \boldsymbol{b} + (\epsilon/\Omega) U \boldsymbol{b} \wedge (\boldsymbol{b} \cdot \nabla) \boldsymbol{b},$$

$$\boldsymbol{b}^{**} = \boldsymbol{b}^* + \frac{\epsilon}{\Omega} \frac{1}{mU} \left[\left(mW^2/2B \right) \boldsymbol{b} \wedge \nabla B + e \boldsymbol{b} \wedge \nabla \Phi \right],$$

$$B_{\|}^* = B[1 + (\epsilon/\Omega) U \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})],$$

$$\boldsymbol{R} = \nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \nabla \boldsymbol{e}_1 \cdot \boldsymbol{e}_2.$$

It is sometimes convenient to define a "modified magnetic field" $\boldsymbol{B}^*$,

$$\boldsymbol{B}^* = \boldsymbol{B} + \frac{\epsilon}{\Omega} BU(\nabla \wedge \boldsymbol{b}). \tag{8.6}$$

This quantity is useful in writing the equations in compact form and in simplifying some proofs (as will be seen later). We note the property

$$\nabla \cdot \boldsymbol{B}^* = 0. \tag{8.7}$$

The modified field is used for constructing three auxiliary quantities. We first define the scalar $B_\parallel^*$ as the projection of $\boldsymbol{B}^*$ on the true magnetic field $\boldsymbol{B}$,

$$B_\parallel^* = \boldsymbol{B}^* \cdot \boldsymbol{b} = B\left(1 + \frac{\epsilon}{\Omega} U \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})\right). \tag{8.8}$$

Next, we introduce the vector $\boldsymbol{b}^*$,

$$\boldsymbol{b}^* = \frac{\boldsymbol{B}^*}{B_\parallel^*} = \boldsymbol{b} + \frac{\epsilon}{\Omega} U \, \boldsymbol{b} \wedge (\boldsymbol{b} \cdot \nabla)\boldsymbol{b}. \tag{8.9}$$

This form follows from the identity

$$\nabla \wedge \boldsymbol{b} - \boldsymbol{b}\, \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}) = \boldsymbol{b} \wedge (\boldsymbol{b} \cdot \nabla)\boldsymbol{b}. \tag{8.10}$$

We also define a vector $\boldsymbol{b}^{**}$ as

$$\boldsymbol{b}^{**} = \boldsymbol{b}^* + \frac{\epsilon}{\Omega} \frac{1}{mU} \left(\frac{mW^2}{2B} \boldsymbol{b} \wedge \nabla B + e\boldsymbol{b} \wedge \nabla \Phi \right). \tag{8.11}$$

We note that, through order ϵ, both $\boldsymbol{b}^*$ and $\boldsymbol{b}^{**}$ are unit vectors:

$$\boldsymbol{b}^* \cdot \boldsymbol{b}^* = \boldsymbol{b}^{**} \cdot \boldsymbol{b}^{**} = \boldsymbol{b} \cdot \boldsymbol{b} = 1 + \mathrm{O}(\epsilon^2). \tag{8.12}$$

The Jacobian (2) (table 8.1) has a very simple form in terms of $B_\parallel^*$. This Jacobian is simply calculated from the matrix of the fundamental Lie brackets by using relation (2.36). The reader may appreciate how much more cumbersome would be the direct calculation of this Jacobian starting from eqs. (8.2)–(8.5).

Next, we note that *the Hamiltonian is form-invariant*:

$$H(\boldsymbol{q}, v_\parallel, v_\perp, \varphi) = \tfrac{1}{2} m \left(v_\parallel^2 + v_\perp^2 \right) + e\Phi(\boldsymbol{q})$$

$$\rightarrow H(\boldsymbol{Y}, U, W, \phi) = \tfrac{1}{2} m (U^2 + W^2) + e\Phi(\boldsymbol{Y}) + \mathrm{O}(\epsilon^2), \tag{8.13}$$

We have seen in section 1.7 that this condition can be *imposed* as a limitation on the choice of the averaging variables *. This condition introduces a satisfactory symmetry between U and W.

* Littlejohn's (1983) "standard guiding centre variables" do not have this property.

The equations of motion have a quite compact form when written in terms of the auxiliary field $\boldsymbol{B}^*$. However, their physical analysis is clearer on the expanded forms, which are written here through the highest significant order, i.e. order ϵ for $\dot{Y}$, $\dot{U}$, $\dot{W}$ and order ϵ^0 for $\dot{\phi}$:

$$\dot{\boldsymbol{Y}} = U\boldsymbol{b} + \frac{\epsilon}{\Omega}\left[\left(\frac{W^2}{2B}\right)\boldsymbol{b}\wedge\nabla B + U^2\boldsymbol{b}\wedge(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \frac{e}{m}\boldsymbol{E}\wedge\boldsymbol{b}\right], \tag{8.14}$$

$$\dot{U} = -\frac{W^2}{2B}\boldsymbol{b}\cdot\nabla B + \frac{e}{m}\boldsymbol{E}\cdot\boldsymbol{b} - \frac{\epsilon}{\Omega}\frac{U}{m}[\boldsymbol{b}\wedge(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\left(\frac{mW^2}{2B}\nabla B - e\boldsymbol{E}\right), \tag{8.15}$$

$$\dot{W} = \frac{UW}{2B}\boldsymbol{b}\cdot\nabla B + \frac{\epsilon}{\Omega}\frac{W}{2B}\left(U^2[\boldsymbol{b}\wedge(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}] - \frac{e}{m}(\boldsymbol{b}\wedge\boldsymbol{E})\right)\cdot\nabla B, \tag{8.16}$$

$$\dot{\phi} = \frac{1}{\epsilon}\Omega + U\boldsymbol{b}\cdot\boldsymbol{R} - \tfrac{1}{2}U\boldsymbol{b}\cdot(\nabla\wedge\boldsymbol{b}). \tag{8.17}$$

Equation (8.14) is particularly clear, when compared to (6.37), which was obtained by the method of the average. We see that the perpendicular drift velocity is identical in the two equations (to order ϵ). We recognize the familiar electric drift, the grad-B drift and the centrifugal drift discussed in sections 1.5 and 1.6. But the definition of the parallel motion is now much clearer. Indeed, (8.14) implies that U is simply the parallel component of the guiding centre velocity. It differs from the corresponding term in (6.37), as can be seen from the definition of U, eq. (8.3). This clearly settles the question of the "parallel drift" which was an object of polemics in the older literature (see Wimmel 1982).

Equation (8.15) defines the parallel acceleration, which is due not only to the parallel component of the electric field as in the homogeneous case (7.17), but also to the gradient of B and to centrifugal effects. The motion of the "true" gyrophase ϕ is modified by terms of order ϵ, due to curvature effects, as appears from (8.17). A discussion of these effects can be found in the paper by Littlejohn (1981). The variable W does not have a direct simple interpretation: its meaning will become clearer after the forthcoming discussion.

At the beginning of section 1.7 it was shown that the pseudocanonical averaging transformation is not unique. Having found one set Z^k of averaging variables (such as Y, U, W, ϕ), any set of functions $\hat{Z}^k$ of these variables, defined by (7.5), is again a set of averaging variables (i.e. it has non-oscillating

fundamental Lie brackets). Among all possible sets of such variables, we define two classes which have a clear physical meaning.

First, we may transform the coordinates Y_1, Y_2, Y_3 among themselves, leaving U, W, ϕ unchanged:

$$Y_i' = Y_i'(Y_1, Y_2, Y_3). \tag{8.18}$$

This corresponds to transforming the Cartesian coordinates to some other coordinate system for the description of the guiding centre position. Such transformations are very important in the study of toroidal confinement systems (see chapters 9, 14 and 16).

On the other hand, the variables U, W may be replaced by new functions P_1, P_2 of $\boldsymbol{Y}$ (or $\boldsymbol{Y}'$), U and W, leaving $\boldsymbol{Y}$ (or $\boldsymbol{Y}'$) unchanged.

We will call *guiding centre variables* any set of averaging variables $(\boldsymbol{Y}, P_1, P_2, \phi)$ for which $\boldsymbol{Y}$ are the (Cartesian or non-Cartesian) coordinates of the guiding centre. Among all sets of guiding centre variables, a convenient subset is the one which leaves the Hamiltonian form-invariant, as in (8.13). In the general case of variables $(P_1, P_2) \neq (U, W)$, this property is defined by requiring the Hamiltonian, expressed in terms of the new variables, to contain no term of order ϵ. Guiding centre variables which leave the Hamiltonian form-invariant will be called *Natural Guiding Centre variables*, or briefly, *NGC-variables*.

Among the possible variables describing a mechanical system, a particular relevance attaches to the *invariants of motion*, both for their physical importance, and for the simplifications they introduce in the description. One of these invariants is obvious: it is the *total energy* of the particle,

$$\mathscr{E} = \tfrac{1}{2}m(U^2 + W^2) + e\Phi(\boldsymbol{Y}). \tag{8.19}$$

Less obvious, but equally important is the following quantity, which will be called the *magnetic moment*:

$$M = \frac{m}{2}\frac{W^2}{B(\boldsymbol{Y})}. \tag{8.20}$$

Clearly, for $\epsilon \to 0$, the quantity reduces to the magnetic moment μ of the particle, defined in (5.12). The invariance of M is easily proved by using eqs. (1) and (3) in table 8.1:

$$\dot{M} = [M, H] = 0 + O(\epsilon^2). \tag{8.21}$$

This property helps interpreting the variable W through its relation to the invariant M,

$$W = +\sqrt{\tfrac{1}{2}mB(Y)\ M}\,. \tag{8.22}$$

Let us stress the fact that in this inversion of (8.20), *the sign of the square root is necessarily positive*. This is because W must tend continuously towards the non-negative variable $v_\perp$ as $\epsilon \to 0$ (see 4.15).

We emphasize the fact that the two invariants $\mathscr{E}$ and M do not have the same status. The energy is an *absolute invariant*, i.e. an invariant of the exact equations of motion in table 6.1,

$$\left[\tfrac{1}{2}m\left(v_\parallel^2 + v_\perp^2\right) + e\Phi(\boldsymbol{q}),\ H(\boldsymbol{q}, v_\parallel, v_\perp)\right] = 0. \tag{8.23}$$

On the contrary, the magnetic moment is *not* an invariant of the exact equations of motion. From table 6.1 we find

$$\dot{\mu} = \left[\frac{m}{2B(\boldsymbol{q})}v_\perp^2\,,\ H(\boldsymbol{q}, v_\parallel, v_\perp)\right]$$

$$= m\frac{v_\perp}{B}\left(-v_\parallel \boldsymbol{n}_2 \cdot \boldsymbol{D} - \frac{e}{m}\boldsymbol{n}_1 \cdot \nabla\Phi\right) - \frac{m}{2}\frac{v_\perp^2}{B^2}\left(v_\parallel \boldsymbol{b} + v_\perp \boldsymbol{n}_1\right)\cdot \nabla B. \tag{8.24}$$

We see that μ *is a constant of the motion when B is homogeneous and there is no electric field*. The question arises whether this invariant can be extended into a new quantity which remains invariant when the magnetic field and the electric potential vary slowly. If such a quantity exists, it is called an *adiabatic invariant of motion*. The result (8.21) shows that the averaging pseudo-canonical transformation realizes precisely this continuation of the magnetic moment μ into an adiabatic invariant M, to order ϵ.

Let us clarify this important point. Suppose we have constructed explicitly the averaging transformation to one higher order, i.e. the Lie brackets of the extended variables are free from oscillations through order ϵ^2. It is then possible to extend the definition of the magnetic moment invariant M (by adding an ϵ^2-correction) in order to achieve the relation $\dot{M} = 0 + O(\epsilon^3)$. The existence of the magnetic moment invariant, constant to *all* orders in ϵ was proved mathematically by Kruskal (1957) [but its explicit form is known only through order ϵ^2: Gardner (1966) for an axisymmetric field geometry and $E = 0$; Weyssow and Balescu (1986) for the general case].

The quantities $\mathscr{E}$ and M are quite convenient for constructing new sets of NGC-variables. An interesting variable, that will prove quite useful in the neoclassical transport theory, is the *kinetic energy* K,

$$K = \mathscr{E} - e\Phi(\boldsymbol{Y}). \tag{8.25}$$

Table 8.2
Natural guiding centre variables $\boldsymbol{Y}$, U, M, ϕ.

Definition

$$M = \frac{m}{2}\frac{W^2}{B(\boldsymbol{Y})}.$$

Fundamental Lie brackets

$$[Y_i, Y_j] = -\epsilon\frac{1}{m\Omega}\epsilon_{ijk}b_k, \qquad [\boldsymbol{Y}, U] = m^{-1}\boldsymbol{b}^*, \qquad [\boldsymbol{Y}, M] = 0,$$

$$[\boldsymbol{Y}, \phi] = \epsilon\frac{1}{m\Omega}\boldsymbol{b}\wedge\boldsymbol{R}, \qquad [U, M] = 0.$$

$$[U, \phi] = -\frac{1}{m}\boldsymbol{b}^*\cdot\boldsymbol{R} + \frac{1}{2m}\boldsymbol{b}\cdot(\nabla\wedge\boldsymbol{b}), \qquad [M, \phi] = -\frac{1}{\epsilon}\frac{\Omega}{B}. \tag{1}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{v}) \to (\boldsymbol{Y}, U, M, \phi)$

$$|J| = m^{-1}B_{\|}^*. \tag{2}$$

Hamiltonian

$$H = (m/2)U^2 + MB(\boldsymbol{Y}) + e\Phi(\boldsymbol{Y}). \tag{3}$$

Equations of motion

$$\dot{\boldsymbol{Y}} = U\boldsymbol{b}^{**}, \tag{4}$$

$$\dot{U} = -m^{-1}\boldsymbol{b}^{**}\cdot\nabla(MB + e\Phi), \tag{5}$$

$$\dot{M} = 0, \tag{6}$$

$$\dot{\phi} = \epsilon^{-1}\Omega + U\boldsymbol{b}\cdot\boldsymbol{R} - \tfrac{1}{2}U\boldsymbol{b}\cdot(\nabla\wedge\boldsymbol{b}). \tag{7}$$

Remarks

All fields Ω, B, Φ, $\boldsymbol{b}$, $\boldsymbol{b}^*$, $\boldsymbol{b}^{**}$ are evaluated at $\boldsymbol{x} = \boldsymbol{Y}$; $\nabla = \partial/\partial\boldsymbol{Y}$,

$$\boldsymbol{b}^* = \boldsymbol{b} + (\epsilon/\Omega)U\boldsymbol{b}\wedge(\boldsymbol{b}\cdot\nabla)\boldsymbol{b},$$

$$\boldsymbol{b}^{**} = \boldsymbol{b}^* + \frac{\epsilon}{\Omega}\frac{1}{mU}\boldsymbol{b}\wedge\nabla(MB + e\Phi),$$

$$B_{\|}^* = B[1 + (\epsilon/\Omega)U\boldsymbol{b}\cdot(\nabla\wedge\boldsymbol{b})],$$

$$\boldsymbol{R} = \nabla\boldsymbol{n}_1\cdot\boldsymbol{n}_2 = \nabla\boldsymbol{e}_1\cdot\boldsymbol{e}_2.$$

Table 8.3
Natural guiding centre variables Y, $\mathscr{E}$, M, ϕ.

Definition

$$\mathscr{E} = (m/2)(U^2 + W^2) + e\Phi(Y), \qquad M = \left[mW^2/2B(Y)\right].$$

Fundamental Lie brackets

$$[Y_i, Y_j] = -\epsilon \frac{1}{m\Omega} \epsilon_{ijk} b_k, \qquad [Y, \mathscr{E}] = Ub^{**}, \qquad [Y, M] = 0$$

$$[Y, \phi] = \epsilon \frac{1}{m\Omega} (\boldsymbol{b} \wedge \boldsymbol{R}), \qquad [\mathscr{E}, M] = 0,$$

$$[\mathscr{E}, \phi] = -\epsilon^{-1}\Omega - Ub \cdot \boldsymbol{R} + \tfrac{1}{2} Ub \cdot (\nabla \wedge \boldsymbol{b}), \qquad [M, \phi] = -\frac{1}{\epsilon} \frac{\Omega}{B}. \tag{1}$$

Jacobian $(q, v) \to (Y, \mathscr{E}, M, \phi)$

$$|J| = \frac{B_{\|}^*}{m^2 |U|}. \tag{2}$$

Hamiltonian

$$H = \mathscr{E}. \tag{3}$$

Equations of motion

$$\dot{Y} = Ub^{**}, \tag{4}$$

$$\dot{\mathscr{E}} = 0, \tag{5}$$

$$\dot{M} = 0, \tag{6}$$

$$\dot{\phi} = \epsilon^{-1}\Omega + Ub \cdot \boldsymbol{R} - \tfrac{1}{2} Ub \cdot (\nabla \wedge \boldsymbol{b}). \tag{7}$$

Remarks

All fields Ω, B, Φ, $B_{\|}^*$, $\boldsymbol{b}$, $\boldsymbol{b}^{**}$ are evaluated at $\boldsymbol{x} = \boldsymbol{Y}$: $\nabla = \partial/\partial \boldsymbol{Y}$,

$$U = \sigma \sqrt{(2/m)(\mathscr{E} - e\Phi - MB)}, \qquad \sigma = \pm,$$

$$\boldsymbol{b}^{**} = \boldsymbol{b} + \frac{\epsilon}{\Omega} \frac{1}{mU} \boldsymbol{b} \wedge \left[mU^2 (\boldsymbol{b} \cdot \nabla) \boldsymbol{b} + \nabla (MB + e\Phi) \right],$$

$$B_{\|}^* = B[1 + (\epsilon/\Omega) Ub \cdot (\nabla \wedge \boldsymbol{b})],$$

$$\boldsymbol{R} = \nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \nabla \boldsymbol{e}_1 \cdot \boldsymbol{e}_2.$$

Among the possible combinations of these NGC variables, the following ones are particularly useful in practice:

$$\{Y, U, M, \phi\}, \qquad \{Y, \mathscr{E}, M, \phi\}, \qquad \{Y, K, M, \phi\}.$$

Table 8.4
Natural guiding centre variables $\boldsymbol{Y}$, K, M, ϕ.

Definition

$$K = (m/2)(U^2 + W^2), \qquad M = \left[mW^2/2B(\boldsymbol{Y})\right].$$

Fundamental Lie brackets

$$\left[Y_i, Y_j\right] = -\epsilon \frac{1}{m\Omega} \varepsilon_{ijk} b_k, \qquad [\boldsymbol{Y}, K] = U\boldsymbol{b}^{**}, \qquad [\boldsymbol{Y}, M] = 0$$

$$[\boldsymbol{Y}, \phi] = \epsilon \frac{1}{m\Omega} (\boldsymbol{b} \wedge \boldsymbol{R}), \qquad [K, M] = 0,$$

$$[K, \phi] = -\epsilon^{-1}\Omega - U\boldsymbol{b} \cdot \boldsymbol{R} + \tfrac{1}{2} U\boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}), \qquad [M, \phi] = -\frac{1}{\epsilon} \frac{\Omega}{B}. \tag{1}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{v}) \to (\boldsymbol{Y}, K, M, \phi)$

$$|J| = \frac{B_{\|}^*}{m^2 |U|}. \tag{2}$$

Hamiltonian

$$H = K + e\Phi(\boldsymbol{Y}). \tag{3}$$

Equations of motion

$$\dot{\boldsymbol{Y}} = U\boldsymbol{b}^{**}, \tag{4}$$

$$\dot{K} = eU\boldsymbol{b}^{**} \cdot \boldsymbol{E}, \tag{5}$$

$$\dot{M} = 0, \tag{6}$$

$$\dot{\phi} = \epsilon^{-1}\Omega + U\boldsymbol{b} \cdot \boldsymbol{R} - \tfrac{1}{2} U\boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}). \tag{7}$$

Remarks

All fields Ω, B, Φ, $B_{\|}^*$, $\boldsymbol{b}$, $\boldsymbol{b}^{**}$ are evaluated at $\boldsymbol{x} = \boldsymbol{Y}$; $\nabla = \partial/\partial \boldsymbol{Y}$,

$$U = \sigma \sqrt{(2/m)\{K - MB\}}, \qquad \sigma = \pm,$$

$$\boldsymbol{b}^{**} = \boldsymbol{b} + \frac{\epsilon}{\Omega} \frac{1}{mU} \boldsymbol{b} \wedge \left[mU^2 (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + \nabla(MB + e\Phi)\right],$$

$$B_{\|}^* = B\left[1 + (\epsilon/\Omega) U\boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})\right],$$

$$\boldsymbol{R} = \nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \nabla \boldsymbol{e}_1 \cdot \boldsymbol{e}_2.$$

The relevant properties of these NGC-variables are collected in tables 8.2–8.4.

The choise $\{\boldsymbol{Y}, \mathscr{E}, M, \phi\}$ is, of course, the one which leads to the simplest equations of motion: two of these reduce to triviality. We may note the following important point. Whenever this set is used, the parallel guiding

centre velocity U must be understood as a function of the basic NGC-variables, defined from (8.19) and (8.20),

$$U \equiv U(\boldsymbol{Y}, \mathscr{E}, M) = \sigma\sqrt{(2/m)[\mathscr{E} - e\Phi(\boldsymbol{Y}) - MB(\boldsymbol{Y})]}\,, \quad \sigma = \pm\,. \tag{8.26}$$

Contrary to W, (8.22), the quantity U may be positive or negative (see 4.14). The *sign factor* $\sigma = \pm 1$ in this equation is therefore important in the inversion of (8.19), and its correct determination must always be considered carefully, as will be seen in forthcoming applications (see, e.g. chapter 9 and 14).

1.9. The drift approximation: The averaging pseudo-canonical transformation. III. Slowly time-dependent, inhomogeneous fields

We now lift the restriction made in the previous sections and allow for a time dependence of the magnetic and electric fields. It is, indeed, important to develop tools for treating these *non-autonomous systems*, not only for aesthetical reasons. In most realistic cases of interest in fusion physics, as well as in astrophysics, the electromagnetic fields interacting with the plasma are time-dependent. In one class of problems, the time-dependence is "fast" (compared to appropriate standards). Examples are the problems of plasmas interacting with electromagnetic waves (such as laser beams) or with high-frequency collective modes produced inside the plasma itself. These problems will not be considered in the present volume, as they are typically relevant for the "anomalous" transport. Here, we shall limit ourselves to the case where the time-dependence of the external fields is "slow". The problem then becomes a natural extension of the *drift approximation*: the temporal rate of change of the fields is measured by some power of the drift parameter ϵ *.

The dynamics of systems with time-dependent external parameters can be cast into the same Hamiltonian formalism as described in section 1.2, by using the known formal procedure (Goldstein 1980, Kruskal 1962, Littlejohn 1981)

* It may be mentioned here that the method of pseudo-canonical averaging transformations can also be applied to the motion of a charged particle in presence of a *high frequency* electromagnetic field. The goal to be reached is to find "*oscillation centre*" coordinates whose equations of motion do not contain the fast oscillations induced by the external field. This leads to the well-known problem of the "*ponderomotive effects*" (Weyssow and Balescu 1987). If a strong quasi-static magnetic field is also present, a pseudo-canonical transformation can be constructed, that averages out *both* the fast external frequency and the fast gyration (Weyssow and Balescu 1988). (Additional references will be found in these two papers.)

of the *extended phase space*. It consists of *considering the time t as a dynamical variable* (rather than as an external parameter as in section 1.2), and adjoining to it a canonically conjugate variable h, which is none other than the *total energy* (up to the sign). Thus, the phase space of a single particle is now an eight-dimensional space, spanned by the coordinates $(q_1, q_2, q_3, t; p_1, p_2, p_3, h)$. We stress the fact that these variables must be treated mathematically as *independent* (although physically they are not!).

The dynamical algebra is determined by the specification of the table of fundamental Lie brackets, which extends eqs. (1) in table 4.1:

$$[q_i, q_j] = 0, \qquad [p_i, p_j] = 0, \qquad [q_i, p_j] = \delta_{ij},$$

$$[t, q_i] = 0, \qquad [t, p_i] = 0, \qquad [h, q_i] = 0,$$

$$[h, p_i] = 0, \qquad [t, h] = 1. \tag{9.1}$$

The motion is now parametrized by an abstract "*proper time*" τ. In the forthcoming formulae, the usual overdot notation must be understood as

$$\dot{a} \equiv \frac{\mathrm{d}a(\tau)}{\mathrm{d}\tau}.$$

The equations of motion are governed by an "*extended Hamiltonian*" $\tilde{H}(\boldsymbol{q}, \boldsymbol{p}, t, h)$, which is simply related to the ordinary Hamiltonian (4.3) (which now involves time-dependent potentials),

$$\tilde{H}(\boldsymbol{q}, \boldsymbol{p}, t, h) \equiv H + h = \frac{1}{2m}\left|\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{q}, t)\right|^2 + e\Phi(\boldsymbol{q}, t) + h. \tag{9.2}$$

The equations of motion for the non-autonomous system then take the same form as eq. (2.19),

$$\dot{a} \equiv \frac{\mathrm{d}a}{\mathrm{d}\tau} = [a, \tilde{H}], \tag{9.3}$$

where a is any dynamical function of $\boldsymbol{q}, \boldsymbol{p}, t, h$. From here on, the formalism of Hamiltonian mechanics proceeds in exactly the same form as for autonomous systems. The only problem left is a problem of *interpretation* of the final results. Some procedure must be devised for the elimination, *in the final stage*, of the artificially added variables τ and h. This aspect will be postponed

to section 2.3 *, where it will be shown how the physically relevant quantities can be expressed in terms of physical variables alone.

We now perform a first pseudo-canonical transformation from the canonical momentum $\boldsymbol{p}$ to the velocity $\boldsymbol{v}$, as in table 4.2; $(\boldsymbol{q}, \boldsymbol{p}, t, h) \rightarrow (\boldsymbol{q}, \boldsymbol{v}, t, h)$. The following additional Lie brackets must be adjoined to those of table 4.2:

$$[q_i, t] = 0, \qquad [q_i, h] = 0, \qquad [v_i, t] = 0.$$

$$[v_i, h] = -\frac{e}{mc}\frac{\partial A_i}{\partial t}, \qquad [t, h] = 1. \tag{9.4}$$

The transformed Hamiltonian is

$$\tilde{H}(\boldsymbol{q}, \boldsymbol{v}, t, h) = \tfrac{1}{2}mv^2 + e\Phi(\boldsymbol{q}, t) + h \tag{9.5}$$

and the equations of motion are

$$\dot{\boldsymbol{q}} = \boldsymbol{v}, \qquad \dot{\boldsymbol{v}} = \frac{e}{mc}(\boldsymbol{v} \wedge \boldsymbol{B}) - \frac{e}{m}\left(\nabla\Phi + \frac{1}{c}\partial_t \boldsymbol{A}\right),$$

$$\dot{h} = \frac{e}{c}\boldsymbol{v} \cdot \partial_t \boldsymbol{A} - e\partial_t \Phi, \qquad \dot{t} = 1. \tag{9.6}$$

We note that the two first equations are actually identical to eqs. (5) and (6) in table 4.2. Indeed, we recall that the electric field in the *non-stationary case* is given by

$$\boldsymbol{E}(\boldsymbol{q}, t) = -\nabla\Phi(\boldsymbol{q}, t) - \frac{1}{c}\partial_t \boldsymbol{A}(\boldsymbol{q}, t). \tag{9.7}$$

We now go over to the local cylindrical velocity variables,

$$\boldsymbol{v} = v_{\parallel}\boldsymbol{b}(\boldsymbol{q}, t) + v_{\perp}\boldsymbol{n}_1(\boldsymbol{q}, \varphi, t), \tag{9.8}$$

where the *moving local reference frame* $\boldsymbol{b}$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ is defined as in (4.11), but with time-dependent unit vectors $\boldsymbol{e}_1(\boldsymbol{q}, t)$, $\boldsymbol{e}_2(\boldsymbol{q}, t)$. There is a constraint on the time-dependence of these vectors, which comes from the requirement that, at all times, they be a set of mutually orthogonal unit vectors; thus

$$b^2(\boldsymbol{q}, t) = 1, \qquad \boldsymbol{b}(\boldsymbol{q}, t) \cdot \boldsymbol{e}_1(\boldsymbol{q}, t) = 0, \quad \text{etc., for all } t.$$

Hence

$$\boldsymbol{b} \cdot \partial_t \boldsymbol{b} = 0, \qquad \boldsymbol{b} \cdot \partial_t \boldsymbol{e}_1 + \boldsymbol{e}_1 \cdot \partial_t \boldsymbol{b} = 0, \quad \text{etc.}$$

* Whenever reference is made to a section, equation, figure or table in another chapter, the number is preceded by the chapter number in boldface type.

Using these relations, the new fundamental Lie brackets and equations of motion are easily worked out.

We now make the basic *scaling assumption*, generalizing eq. (6.1). The latter equation is maintained in the mathematically equivalent form (used by Gardner 1966 and by Littlejohn 1981)

$$e \to \frac{1}{\epsilon} e, \qquad B \to \epsilon^{\circ} B \tag{9.9}$$

which must be combined with the assumption

$$\Phi \to \epsilon \Phi \tag{9.10}$$

in order to keep the term $e\Phi$ in the Hamiltonian of zeroth order. In order to choose a reasonable scaling for the rates of change in time, we consider specifically the situation arising in a toroidal confinement system, such as the tokamak (see chapter 8). It appears that in such a configuration, the non-potential part of the electric field (i.e. the second term in eq. (9.7) is of order ϵ compared to the potential part (Hinton and Hazeltine 1976). In order to scale the terms in agreement with this fact, we assume

$$\begin{aligned}
&\partial_t \boldsymbol{A} \to \epsilon^2 \partial_t \boldsymbol{A}, \\
&\partial_t \boldsymbol{b} \to \epsilon^2 \partial_t \boldsymbol{b}, \\
&\partial_t \boldsymbol{n}_i \to \epsilon^2 \partial_t \boldsymbol{n}_i, \\
&\partial_t \Phi \to \epsilon^3 \partial_t \Phi, \quad i = 1, 2.
\end{aligned} \tag{9.11}$$

The set of fundamental Lie brackets is now easily derived. The "old" brackets are the same as in table 6.1 (but, of course, with all the fields depending on time); to these must be added the following brackets, in which we only keep terms through order ϵ:

$$\begin{aligned}
&[q_i, t] = 0, \qquad [v_{\parallel}, t] = 0, \qquad [v_{\perp}, t] = 0, \qquad [\varphi, t] = 0, \\
&[q_i, h] = 0, \qquad [v_{\parallel}, h] = -\epsilon \frac{e}{mc} \boldsymbol{b} \cdot \partial_t \boldsymbol{A}, \qquad [v_{\perp}, h] = -\epsilon \frac{e}{mc} \boldsymbol{n}_1 \cdot \partial_t \boldsymbol{A}, \\
&[\varphi, h] = \epsilon \frac{e}{mcv_{\perp}} \boldsymbol{n}_2 \cdot \partial_t \boldsymbol{A}, \qquad [t, h] = 1.
\end{aligned} \tag{9.12}$$

The extended Hamiltonian is

$$\tilde{H} = \tfrac{1}{2}m\left(v_{\parallel}^2 + v_{\perp}^2\right) + e\Phi(\boldsymbol{q}, t) + h \tag{9.13}$$

and the equations of motion are

$$\begin{aligned}
&\dot{\boldsymbol{q}} = v_{\parallel}\boldsymbol{b} + v_{\perp}\boldsymbol{n}_1,\\
&\dot{v}_{\parallel} = v_{\perp}\boldsymbol{n}_2 \cdot \boldsymbol{D} - \frac{e}{m}\boldsymbol{b}\cdot\nabla\Phi - \epsilon\frac{e}{mc}\boldsymbol{b}\cdot\partial_t\boldsymbol{A},\\
&\dot{v}_{\perp} = -v_{\parallel}\boldsymbol{n}_2\cdot\boldsymbol{D} - \frac{e}{m}\boldsymbol{n}_1\cdot\nabla\Phi - \epsilon\frac{e}{mc}\boldsymbol{n}_1\cdot\partial_t\boldsymbol{A},\\
&\dot{\varphi} = \frac{1}{\epsilon}\Omega + \boldsymbol{b}\cdot\boldsymbol{D} - \frac{v_{\parallel}}{v_{\perp}}\boldsymbol{n}_1\cdot\boldsymbol{D} + \frac{e}{mv_{\perp}}\boldsymbol{n}_2\cdot\nabla\Phi + \epsilon\frac{e}{mcv_{\perp}}\boldsymbol{n}_2\cdot\partial_t\boldsymbol{A},\\
&\dot{h} = \epsilon\frac{e}{c}\left(v_{\parallel}\boldsymbol{b}\cdot\partial_t\boldsymbol{A} + v_{\perp}\boldsymbol{n}_1\cdot\partial_t\boldsymbol{A}\right),\\
&\dot{t} = 1.
\end{aligned} \tag{9.14}$$

From here on, we shall proceed, as in sections 1.7 and 1.8, to the construction of an *averaging pseudo-canonical transformation*. Let us call $\tilde{Z}^{\lambda}$, $\lambda = 1,\ldots,8$, the extended phase-space variables, with the conventions $\tilde{Z}^6 = \phi$, $\tilde{Z}^7 = k$, $\tilde{Z}^8 = t'$. The extended Kruskal–Littlejohn theorem can be formulated as follows.

There exists a pseudo-canonical transformation from the particle variables $\tilde{z}^{\lambda}$ to a set of a new phase-space variables $\tilde{Z}^{\lambda}$, such that all the fundamental Lie brackets of the new variables $\tilde{\Sigma}^{\lambda\mu} \equiv [\tilde{Z}^{\lambda}, \tilde{Z}^{\mu}]$ are independent of $\tilde{Z}^6 \equiv \phi$.

The proof is a constructive one. We write

$$\tilde{Z}^{\lambda} = \tilde{z}^{\lambda} + \frac{\epsilon}{\Omega}\left[\zeta_0^{\lambda}(\tilde{z}^{\mu}) + \zeta_{\mathrm{T}}^{\lambda}(\tilde{z}^{\mu})\right] + \mathrm{O}(\epsilon^2), \tag{9.15}$$

where ζ_0^{λ} defines the pseudo-canonical transformation in the time-independent case, and the functions $\zeta_{\mathrm{T}}^{\lambda}$ are the additional terms to be added when the fields are non-stationary. Thus, ζ_0^{λ} are defined by eqs. (8.2)–(8.5) for $\lambda = 1,\ldots,6$, and $\zeta_0^7 = \zeta_0^8 = 0$. The functions $\zeta_{\mathrm{T}}^{\lambda}$ are determined in such a way as to annul all the gyrophase-dependent terms in the Lie brackets of the new variables. We

omit again the detailed calculations and quote directly the final result *through order* ϵ *, choosing the most useful set of NGC-variables, Y, $\mathscr{E}$, M, ϕ, k, t',

$$Y = q - \frac{\epsilon}{\Omega} v_\perp n_2,$$

$$\mathscr{E} = \tfrac{1}{2} m \left(v_\parallel^2 + v_\perp^2 \right) + e\Phi,$$

$$M = \frac{m}{2B} v_\perp^2 + \frac{\epsilon}{\Omega} \frac{m}{B} \times \Big(v_\parallel^2 v_\perp n_1 \cdot (\nabla \wedge b) - \tfrac{1}{4} v_\parallel v_\perp^2 \left[n_2 \cdot (\nabla \wedge n_2) - n_1 \cdot (\nabla \wedge n_1) - 2b \cdot (\nabla \wedge b) \right] - \frac{v_\perp^3}{2B} n_1 \cdot \nabla B - \frac{e}{m} v_\perp n_2 \cdot \nabla \Phi \Big),$$

$$\phi = \varphi + \frac{\epsilon}{\Omega} \Big(\frac{v_\parallel^2}{v_\perp} n_2 \cdot (\nabla \wedge b) - v_\perp b \cdot (\nabla \wedge n_2) + \tfrac{1}{4} v_\parallel \left[n_2 \cdot (\nabla \wedge n_1) + n_1 \cdot (\nabla \wedge n_2) \right] + v_\perp B^{-1} n_1 \cdot \nabla B + \frac{e}{m v_\perp} n_1 \cdot \nabla \Phi \Big),$$

$$k = h, \qquad t' = t. \tag{9.16}$$

We note that, through order ϵ, all the coefficients ζ_Γ^λ are null. Expressions (9.16) of the NGC-variables coincide with those derived from (8.2)–(8.5).

The extended Hamiltonian has the very simple form

$$\tilde{H} = \mathscr{E} + k. \tag{9.17}$$

As usual, we collect the fundamental Lie brackets, the Jacobian and the equations of motion in table 9.1. All these results are correct through order ϵ (ϵ^0 for the brackets involving ϕ and for $\dot{\phi}$). We note that, due to the ordering (9.11), the results are very simple. The only modified equations of motion are

* The terms of order ϵ^2 were obtained by Weyssow and Balescu (1986).

Table 9.1
Natural guiding centre variables $\boldsymbol{Y}$, $\mathscr{E}$, M, ϕ, k, t.

Definition
See eqs. (9.16).

Fundamental Lie brackets

$$[Y_i, Y_j] = -\epsilon \frac{1}{m\Omega} \varepsilon_{ijk} b_k, \qquad [\boldsymbol{Y}, \mathscr{E}] = U\boldsymbol{b}^{**}, \qquad [\boldsymbol{Y}, M] = 0,$$

$$[\boldsymbol{Y}, \phi] = \epsilon \frac{1}{m\Omega} \boldsymbol{b} \wedge \boldsymbol{R}, \qquad [\boldsymbol{Y}, k] = 0, \qquad [\boldsymbol{Y}, t] = 0,$$

$$[\mathscr{E}, M] = 0, \qquad [\mathscr{E}, \phi] = -\epsilon^{-1}\Omega - U\boldsymbol{b} \cdot \boldsymbol{R} + \tfrac{1}{2} U\boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}),$$

$$[\mathscr{E}, k] = -\epsilon \frac{e}{c} U\boldsymbol{b} \cdot \partial_t \boldsymbol{A}, \qquad [\mathscr{E}, t] = 0, \qquad [M, k] = 0,$$

$$[M, \phi] = -\frac{1}{\epsilon} \frac{\Omega}{B}, \qquad [M, t] = 0, \qquad [\phi, k] = 0,$$

$$[\phi, t] = 0, \qquad [k, t] = -1. \tag{1}$$

Jacobian $(\boldsymbol{q}, v, h, t) \to (\boldsymbol{Y}, \mathscr{E}, M, \phi, k, t)$

$$|J| = \frac{B_{\|}^*}{m^2 |U|} \tag{2}$$

Hamiltonian

$$\tilde{H} = \mathscr{E} + k \tag{3}$$

Equations of motion

$$\dot{\boldsymbol{Y}} = U\boldsymbol{b}^{**}, \tag{4}$$

$$\dot{\mathscr{E}} = -\epsilon (e/c) U\boldsymbol{b} \cdot \partial_t \boldsymbol{A}, \tag{5}$$

$$\dot{M} = 0, \tag{6}$$

$$\dot{\phi} = \epsilon^{-1}\Omega + U\boldsymbol{b} \cdot \boldsymbol{R} - \tfrac{1}{2} U\boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}), \tag{7}$$

$$\dot{k} = \epsilon (e/c) U\boldsymbol{b} \cdot \partial_t \boldsymbol{A}, \tag{8}$$

$$\dot{t} = 1. \tag{9}$$

Remarks
All fields Ω, B, Φ, $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{b}^{**}$, $B_{\|}^*$ are evaluated at $\boldsymbol{x} = \boldsymbol{Y}$; $\nabla = \partial/\partial \boldsymbol{Y}$,

$$U = \sigma \sqrt{(2/m)(\mathscr{E} - e\Phi - MB)}, \qquad \sigma = \pm,$$

$$\boldsymbol{b}^{**} = \boldsymbol{b} + \frac{\epsilon}{\Omega m U} \boldsymbol{b} \wedge \left[mU^2 (\boldsymbol{b} \cdot \nabla) \boldsymbol{b} + M \nabla B + e \nabla \Phi \right],$$

$$B_{\|}^* = B[1 + (\epsilon/\Omega) U\boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})],$$

$$\boldsymbol{R} = \nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \nabla \boldsymbol{e}_1 \cdot \boldsymbol{e}_2.$$

those for $\dot{\mathscr{E}}$ and $\dot{k}$. Indeed, *the energy is no longer a constant of the motion* in a time-dependent external field. It is important to emphasize, however, that *the magnetic moment M is still an adiabatic invariant of the motion*. Next, we note that the right-hand sides of the equations for $\dot{\mathscr{E}}$ and $\dot{k}$ are identical (up to the sign): this is consistent with the fact that the "artificial" phase-space variable k must be physically identified with (minus) the energy. Also, the equation of motion $\dot{t} = 1$ allows the physical identification of t with the "proper time" τ. Some care must, however, be taken in the expression of this final identification, as will be shown in section **2.3**.

With these results we have completed the purely mechanical study of the motion of a single particle in a general, weakly inhomogeneous, slowly varying electromagnetic field. We shall come back to this problem in chapter 9, where the present general results will be applied to the motion of a particle in a specific, toroidal geometry of the magnetic field.

The next chapter is devoted to the passage from the individual particle picture to the picture of a *plasma*, considered as a system of many charged particles, interacting with each other.

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2

The microscopic description of a plasma

2.1. *Statistical description of a plasma*

Having discussed at length the dynamics of a single particle, we now turn our attention to a *system of N particles*, where N is a very large number ($N \approx 10^{23}$). It is well known that very large systems cannot be treated by the methods of ordinary mechanics: we enter the realm of *statistical mechanics*. It is, of course, out of the question to develop in the present book the details of statistical mechanics and kinetic theory. Many books on this subject exist on the market (e.g. Balescu 1963, 1975, Klimontovich 1964, 1982, Akhiezer and Peletminskii 1981, Lifshitz and Pitaevskii 1981). We assume the reader to be familiar with the elements of statistical physics; we shall provide, however, a sketchy derivation of the main results.

In the present section we shall fix the definition of a quiescent plasma to be used in this book and discuss the general framework of the theory. A definition is necessarily a limitation of a vast subject. Our purpose is to concentrate on the role of the *electromagnetic interactions* of charged particles among themselves and/or with external fields. We therefore restrict ourselves to the study of *fully ionized plasmas*. This, of course, excludes a number of very interesting phenomena, which are quite important in partially ionized plasmas. The latter contain a fraction of neutral molecules and/or atoms, which may undergo a number of transformations under the action of collisions or of the external field. The study of such processes as excitation or deexcitation, ionization or recombination of the atoms or ions requires, beyond the methods of kinetic theory, the "full artillery" of atomic physics. The problem becomes very difficult on the level of a minimally rigorous theory.

We define a *(fully ionized) plasma* as a collection of electrons and of positively charged ions.

The nature of the ions can be quite variable. In plasmas of thermonuclear interest, as well as in many astrophysical applications, the ions are nuclei of hydrogen, H^+, or of its heavy isotopes, deuterium D^+, or tritium T^+. Many experiments are, however, performed with metals (gold, aluminum, iron, cesium, ...) which are present in the final plasma in various states of ionization; in a magnetically confined plasma there always exists a more or less

important fraction of heavy ion impurities originating from the walls. These examples show that in a detailed description of a fully ionized plasma, one should take into account the presence of a number of different ion species. For simplicity, we shall discuss in this book only such plasmas in which we either have a single species of ions, or in which we can model the real mixture by a single "effective" ion species. A few remarks and references for the problem of multispecies plasmas will be given in chapter 16.

As a consequence of our limitation to fully ionized plasmas (hence of exclusion of internal atomic processes), we may idealize both the electrons and the ions as *point particles, subject to the laws of classical (i.e. non-quantum) mechanics and electrodynamics*. Their microscopic nature is thus fully characterized by two parameters: their *mass* and their *charge*.

We now fix some conventions of notation. Quantities relevant to particles of a given species will be labelled by a greek index taken at the beginning of the alphabet, most usually: α, β, γ,... (it may appear either as a subscript or as a superscript). This index may take two values: $\alpha = \mathrm{e}$ for the electrons and $\alpha = \mathrm{i}$ for the ions. We thus denote by m_α the *mass* of the particles of species α, and by e_α their *charge*. More specifically, if we call e the *absolute value of the charge of the electron* (a fundamental physical constant), we have

$$e_\mathrm{e} = -e, \qquad e_\mathrm{i} = +Ze, \tag{1.1}$$

where Z is the *charge number* of the ions. Throughout this book we use *Gaussian (cgs-) units*, which are more natural in problems of microscopic physics.

Concerning masses, an important fact must already be noted. A characteristic feature of all the electron–ion plasmas is the very large disparity of the masses of the ions and of the electrons. Even in the extreme case of a hydrogen plasma, the electron-to-ion mass ratio is approximately $1/1836 \simeq 5.45 \times 10^{-4}$. The smallness of this ratio will be systematically used in order to simplify the expressions. We thus assume that

$$\mu \equiv \frac{m_\mathrm{e}}{m_\mathrm{i}} \ll 1 \tag{1.2}$$

and we consistently neglect quantities that are of order μ (but not, for example, quantities of order $\mu^{1/2}$).

We denote the total number of particles of species α by the symbol N_α. In a fully ionized plasma, i.e. in the absence of ionization and recombination processes, these numbers are constant in time for each species. Among these numbers exists a relation, expressing that the plasma is *globally neutral*, i.e. the

total negative charge of the electrons exactly compensates the total positive charge of the ions,

$$N_e = ZN_i. \tag{1.3}$$

(This does not preclude the possibility of *local* deviations from electroneutrality, which will be discussed in section **4**.1). We also introduce the notation N for the total number of particles,

$$N = \sum_\alpha N_\alpha. \tag{1.4}$$

Let the index j characterize a given particle (electron or ion) with a range of values 1, 2,..., N. Let the vector $\boldsymbol{q}_j$ denote the coordinate (or position) of particle j, and the vector $\boldsymbol{p}_j$ represent its canonical momentum. The set of quantities $(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_N, \boldsymbol{p}_1,\ldots,\boldsymbol{p}_N)$ determines a point in a $6N$-dimensional space, called the *phase space* of the dynamical system. We shall often use the abbreviation

$$(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_N, \boldsymbol{p}_1,\ldots,\boldsymbol{p}_N) \equiv (q, p). \tag{1.5}$$

Let also $H(q, p)$ be the *Hamiltonian* of the system, which we assume here to be time-independent. In section 2.3 we discuss the necessary adaptation for the treatment of slowly time-dependent external fields.

The specification of a point in phase space completely determines the state of the system in Hamiltonian mechanics (see section **1**.2). However, in dealing with very large systems, such a description is inadequate. We must introduce the tools of statistical mechanics.

It is well known (Balescu 1975) that the statistical state of a system is determined by a *phase-space distribution function* $F(q, p; t)$: a non-negative function of the $6N$ phase-space coordinates and of the time, which is normalized to one, as

$$\int \mathrm{d}^{3N}q\, \mathrm{d}^{3N}p\, F(q, p; t) = 1. \tag{1.6}$$

On the left-hand side we have a $6N$-fold integral over the whole accessible phase space. $F(q, p; t)$ can therefore be interpreted as the probability density for finding the system at time t at the point (q, p) in phase space. The fact that F is normalized to one at all times is a non-trivial consequence of *Liouville's theorem*, which follows from the fact that the volume of any region of phase space is invariant when its constituent points move according to

Hamilton's equations (see e.g. Goldstein 1980). From (1.6) we therefore conclude that the total time derivative of $F(q, p; t)$ vanishes:

$$\frac{\mathrm{d}F(q, p; t)}{\mathrm{d}t} \equiv \frac{\partial F}{\partial t} + \sum_{j=1}^{N} \left(\dot{\boldsymbol{q}}_j \cdot \frac{\partial F}{\partial \boldsymbol{q}_j} + \dot{\boldsymbol{p}}_j \cdot \frac{\partial F}{\partial \boldsymbol{p}_j} \right) = 0.$$

Combining this result with the Hamilton equations in their general form (1.2.19), we find

$$\frac{\partial F}{\partial t} + \sum_{j=1}^{N} \left([\boldsymbol{q}_j, H] \cdot \frac{\partial F}{\partial \boldsymbol{q}_j} + [\boldsymbol{p}_j, H] \cdot \frac{\partial F}{\partial \boldsymbol{p}_j} \right) = 0. \tag{1.7}$$

Using now property (1.2.14) of the Lie brackets, we get

$$\frac{\partial F}{\partial t} + [F, H] = 0,$$

or

$$\frac{\partial F}{\partial t} = [H, F]. \tag{1.8}$$

This is the celebrated *Liouville equation*, the basis of statistical mechanics. The expression on the right-hand side can also be written as a linear operator, the *Liouvillian* L, acting on the distribution function as

$$\partial_t F = LF, \tag{1.9}$$

where we use the obvious abbreviation $\partial_t \equiv \partial/\partial t$, and where L is defined as the differential operator

$$L \equiv \sum_{j=1}^{N} \left([\boldsymbol{q}_j, H] \cdot \frac{\partial}{\partial \boldsymbol{q}_j} + [\boldsymbol{p}_j, H] \cdot \frac{\partial}{\partial \boldsymbol{p}_j} \right). \tag{1.10}$$

The phase-space distribution function contains the maximum amount of information about the system in a statistical mechanical description. However, for all practical purposes, in particular in kinetic theory and in transport theory, most of this information is irrelevant. It is therefore advantageous to describe the system in terms of much simpler quantities, called the *reduced distribution functions* (rdf) $\hat{f}_s^{\alpha_1 \cdots \alpha_s}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_s, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_s; t)$ (Balescu 1975). (The reason for introducing a "hat" over symbols in this notation will appear in section 2.2). These functions are obtained by integration of F over $N - s$

"irrelevant" phase-space coordinates. In a two-component system, the correct combinatorial factor entering the definition of the general s-particle rdf is rather complicated (it depends on the number of electrons and of ions in the group s). We shall therefore only give the explicit definitions for the one- and two-particle distribution functions,

$$\hat{f}_1^{\alpha}(\boldsymbol{q}_1, \boldsymbol{p}_1; t) = N_\alpha \int \mathrm{d}\boldsymbol{q}_2 \, \mathrm{d}\boldsymbol{p}_2 \cdots \mathrm{d}\boldsymbol{q}_N \, \mathrm{d}\boldsymbol{p}_N \, F(q, p; t), \tag{1.11}$$

$$\hat{f}_2^{\alpha\alpha}(\boldsymbol{q}_1, \boldsymbol{p}_1, \boldsymbol{q}_2, \boldsymbol{p}_2; t) = N_\alpha(N_\alpha - 1) \int \mathrm{d}\boldsymbol{q}_3 \, \mathrm{d}\boldsymbol{p}_3 \cdots \mathrm{d}\boldsymbol{q}_N \, \mathrm{d}\boldsymbol{p}_N \, F(q, p; t), \tag{1.12}$$

$$\hat{f}_2^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{p}_1, \boldsymbol{q}_2, \boldsymbol{p}_2; t) = N_\alpha N_\beta \int \mathrm{d}\boldsymbol{q}_3 \, \mathrm{d}\boldsymbol{p}_3 \cdots \mathrm{d}\boldsymbol{q}_N \, \mathrm{d}\boldsymbol{p}_N \, F(q, p; t). \tag{1.13}$$

In the realistic case of very large systems ($N \gg 1$), these definitions, together with (1.6), imply the normalization conditions for the reduced distribution functions,

$$\int \mathrm{d}\boldsymbol{q}_1 \, \mathrm{d}\boldsymbol{p}_1 \, \hat{f}_1^{\alpha}(\boldsymbol{q}_1, \boldsymbol{p}_1; t) = N_\alpha, \tag{1.14}$$

$$\int \mathrm{d}\boldsymbol{q}_1 \, \mathrm{d}\boldsymbol{p}_1 \, \mathrm{d}\boldsymbol{q}_2 \, \mathrm{d}\boldsymbol{p}_2 \, \hat{f}_2^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{p}_1, \boldsymbol{q}_2, \boldsymbol{p}_2; t) = N_\alpha N_\beta. \tag{1.15}$$

The importance of the reduced distribution functions comes from the fact that all important quantities of transport theory are defined as averages of dynamical quantities $b^{\alpha}(\boldsymbol{q}, \boldsymbol{p})$, weighted with $\hat{f}_1^{\alpha}$ (see chapter 3):

$$\langle b \rangle = \sum_{\alpha} \int \mathrm{d}\boldsymbol{q} \, \mathrm{d}\boldsymbol{p} \, b^{\alpha}(\boldsymbol{q}, \boldsymbol{p}) \hat{f}_1^{\alpha}(\boldsymbol{q}, \boldsymbol{p}; t). \tag{1.16}$$

The main goal of kinetic theory is the derivation of a *closed equation of evolution for the reduced distribution function* $\hat{f}_1^{\alpha}(\boldsymbol{q}, \boldsymbol{p}; t)$. It is well known that, in general, such an equation only exists in an approximate sense. The remaining sections of this chapter are devoted to a brief description of this closure problem.

2.2. *Liouville equation for independent particles in stationary external fields*

In a first stage we consider a fictitious system of non-interacting particles. We do this not only for pedagogical reasons of simplicity: we shall see in forthcoming sections that the results obtained here are useful even in some quite important descriptions of the interacting plasmas.

In the idealized system, the charged particles making up the plasma move under the action of an *external* magnetic field $\boldsymbol{B}(\boldsymbol{x})$, derived from a vector potential $\boldsymbol{A}(\boldsymbol{x})$, and of an *external* electric field $\boldsymbol{E}_0(\boldsymbol{x})$, derived from a scalar potential $\Phi_0(\boldsymbol{x})$. In the present section, all these fields are assumed to be time-independent. We stress again that, for the time being, we disregard the action of the fields produced by the charged particles themselves. The Hamiltonian of this ideal system is then similar to (**1**.4.3):

$$H(q,\ p) = \sum_{j=1}^{N} \left(\frac{1}{2m_{\alpha_j}} \left| \boldsymbol{p}_j - c^{-1} e_{\alpha_j} \boldsymbol{A}(\boldsymbol{q}_j) \right|^2 + e_{\alpha_j} \Phi_0(\boldsymbol{q}_j) \right), \tag{2.1}$$

where the index α_j labels the species of particle j. Let us note the following important structural feature. The Hamiltonian is a *sum* of terms, each of which depends on the phase-space variables of a single particle j, of species α_j,

$$H(q,\ p) = \sum_{j=1}^{N} H_{0j}^{\alpha_j} \equiv \sum_{j=1}^{N} H_{0j}^{\alpha_j}(\boldsymbol{q}_j,\ \boldsymbol{p}_j), \tag{2.2}$$

where the function $H_{0j}^{\alpha_j}$ is easily identified from (2.1). This *additive structure* is characteristic of systems of non-interacting particles.

We now consider the *Liouvillian operator* (1.10) for our system. Because of the linearity of this operator (see eq. **1**.2.12) and of the fundamental Lie brackets (**1**.2.16), we find that the Liouvillian has the same additive structure as the Hamiltonian:

$$L = \sum_{j=1}^{N} L_j^{\alpha_j}, \tag{2.3}$$

where

$$L_j^{\alpha_j} = \left[H_{0j}^{\alpha_j}, \ldots \right] \tag{2.4}$$

is an operator involving only the derivatives $\partial/\partial \boldsymbol{q}_j$, $\partial/\partial \boldsymbol{p}_j$, acting on the coordinates of particle j (of species α_j).

In the present case of a system of non-interacting particles, it is trivially simple to derive from the Liouville equation,

$$\partial_t F = \left[\sum_{j=1}^{N} L_j^{\alpha_j} \right] F, \tag{2.5}$$

an exact, closed equation for the reduced one-particle distribution function. Indeed, recalling definition (1.11), we multiply both sides of (2.5) by N_α and integrate over the positions and momenta of all the particles, except those of particle 1 of species α; we find (Balescu 1975)

$$\partial_t \hat{f}_1^\alpha(\boldsymbol{q}_1, \boldsymbol{p}_1; t) = L_1^\alpha \hat{f}_1^\alpha(\boldsymbol{q}_1, \boldsymbol{p}_1; t)$$

because the contributions of all the one-particle Liouvillians other than L_1^α vanish by partial integration.

We now adopt a more compact notation: we drop the index 1 from $\hat{f}_1^\alpha$, it being understood that $\hat{f}^\alpha$ denotes a one-particle reduced distribution function. Moreover, in the text, we shall refer to $\hat{f}^\alpha$ simply as *the distribution function* (of species α). The index 1 is also superfluous on L_1^α and on the variables $\boldsymbol{q}_1$, $\boldsymbol{p}_1$. Thus, we simply write

$$\partial_t \hat{f}^\alpha = L^\alpha \hat{f}^\alpha(\boldsymbol{q}, \boldsymbol{p}; t), \tag{2.6}$$

or, equivalently,

$$\partial_t \hat{f}^\alpha = \left[H_0^\alpha, \hat{f}^\alpha \right], \tag{2.7}$$

or, even more explicitly,

$$\partial_t \hat{f}^\alpha = \left[H_0^\alpha, \boldsymbol{q} \right] \cdot \frac{\partial \hat{f}^\alpha}{\partial \boldsymbol{q}} + \left[H_0^\alpha, \boldsymbol{p} \right] \cdot \frac{\partial \hat{f}^\alpha}{\partial \boldsymbol{q}}, \tag{2.8}$$

and

$$H_0^\alpha = \frac{1}{2m_\alpha} \left| \boldsymbol{p} - \frac{e_\alpha}{c} \boldsymbol{A}(\boldsymbol{q}) \right|^2 + e_\alpha \Phi_0(\boldsymbol{q}). \tag{2.9}$$

The important feature of this result is the following. For a non-interacting system, the reduced one-particle distribution function obeys an equation of evolution which has the same form as the Liouville equation (1.9) in the reduced phase space defined by the position and the momentum of a single particle.

The Liouville equation was derived in the framework of Hamiltonian mechanics by using a set of canonical variables. The discussion in chapter 1 has shown that such variables are inconvenient for electromagnetic systems. The natural variables, which have a simple physical meaning, are not related to $\boldsymbol{q}$ and $\boldsymbol{p}$ by canonical transformations. The formalism of pseudo-canonical transformations developed in section 1.2 allows us, however, to deal very simply with such changes of variables. Indeed, the equation of evolution (2.7) is expressed in terms of a Lie bracket; its form is therefore independent of the choice of the basic variables. Thus, equation (2.7) is valid in *any set of phase-space coordinates*, provided the Lie bracket is calculated in the appropriate way. Only the *explicit form* of the Liouvillian will be different in different sets of coordinates (as illustrated below).

There is, however, a subtle point that should not be missed when dealing with pseudo-canonical transformations. Upon a transformation from $(\boldsymbol{q}, \boldsymbol{p})$ to $(\boldsymbol{Q}, \boldsymbol{P})$, the distribution function transforms as

$$\hat{f}^{\alpha}(\boldsymbol{q}, \boldsymbol{p}; t) \rightarrow \hat{f}^{\alpha}(\boldsymbol{q}(\boldsymbol{Q}, \boldsymbol{P}), \boldsymbol{p}(\boldsymbol{Q}, \boldsymbol{P}); t) \equiv \hat{f}'^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}; t). \tag{2.10}$$

Any dynamical function $b^{\alpha}(\boldsymbol{q}, \boldsymbol{p})$ transforms similarly into a function $b'^{\alpha}(\boldsymbol{Q}, \boldsymbol{P})$; but the basic formula (1.16) for the average becomes, in the new, non-canonical coordinates,

$$\langle b \rangle = \sum_{\alpha} \int \mathrm{d}\boldsymbol{Q}\, \mathrm{d}\boldsymbol{P} \, | J^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}) | \, b'^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}) \hat{f}'^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}; t). \tag{2.11}$$

The new feature is the appearance of the Jacobian J^{α} of the transformation $(\boldsymbol{q}, \boldsymbol{p}) \rightarrow (\boldsymbol{Q}, \boldsymbol{P})$. Hence, this formula can be interpreted as follows. Under the pseudo-canonical transformation $(\boldsymbol{q}, \boldsymbol{p}) \rightarrow (\boldsymbol{Q}, \boldsymbol{P})$, the *true distribution function*, i.e. the weighting function in the statistical average, transforms as

$$\hat{f}^{\alpha}(\boldsymbol{q}, \boldsymbol{p}; t) \rightarrow | J^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}) | \, \hat{f}'^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}; t). \tag{2.12}$$

In particular, it is the function $| J^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}) | \, \hat{f}'^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}; t)$ that is normalized to N_{α} like the initial function $\hat{f}^{\alpha}(\boldsymbol{q}, \boldsymbol{p}; t)$ (eq. 1.14),

$$\int \mathrm{d}\boldsymbol{Q}\, \mathrm{d}\boldsymbol{P} \, | J^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}) | \, \hat{f}'^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}; t) = N_{\alpha}. \tag{2.13}$$

Note that this important point never arises in the "traditional" problems of statistical mechanics, when consideration is limited to canonical transformations. Indeed, for canonical transformations, the Jacobian equals one (see eq. 1.2.22).

It is therefore desirable to derive an equation for the true distribution function $|J^{\alpha}(\boldsymbol{Q}, \boldsymbol{P})|\hat{f}'^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}; t)$. In order to do so, we need the following lemmas.

Consider first the case when the new variables $(\boldsymbol{Q}^{c}, \boldsymbol{P}^{c})$ are canonical. We then easily prove the identity

$$\sum_{s=1}^{f}\left(\frac{\partial}{\partial Q_s^{c}}[H, Q_s^{c}]+\frac{\partial}{\partial P_s^{c}}[H, P_s^{c}]\right)=0. \tag{2.14}$$

Indeed, by Hamilton's equations (1.2.23), the left-hand side is simply

$$\sum_{s=1}^{f}\left(-\frac{\partial}{\partial Q_s^{c}}\frac{\partial}{\partial P_s^{c}}H+\frac{\partial}{\partial P_s^{c}}\frac{\partial}{\partial Q_s^{c}}H\right)\equiv 0.$$

It then follows that the reduced Liouville equation (2.8) can be cast into two, equivalent forms,

$$\partial_t\hat{f}'^{\alpha}=[H_0^{\alpha}, \boldsymbol{Q}^{c}]\cdot\frac{\partial}{\partial \boldsymbol{Q}^{c}}\hat{f}'^{\alpha}+[H_0^{\alpha}, \boldsymbol{P}^{c}]\cdot\frac{\partial}{\partial \boldsymbol{P}^{c}}\hat{f}'^{\alpha},$$

$$\partial_t\hat{f}'^{\alpha}=\frac{\partial}{\partial \boldsymbol{Q}^{c}}\cdot\left([H_0^{\alpha}, \boldsymbol{Q}^{c}]\hat{f}'^{\alpha}\right)+\frac{\partial}{\partial \boldsymbol{P}^{c}}\cdot\left([H_0^{\alpha}, \boldsymbol{P}^{c}]\hat{f}'^{\alpha}\right). \tag{2.15}$$

The generalization of these formulae for pseudo-canonical transformations rests on the following identity, analogous to (2.14):

$$\sum_{s=1}^{f}\left(\frac{\partial}{\partial Q_s}(|J^{\alpha}|[H, Q_s])+\frac{\partial}{\partial P_s}(|J^{\alpha}|[H, P_s])\right)=0. \tag{2.16}$$

This remarkable formula holds for *any Hamiltonian*. Its proof is rather cumbersome and will not be given here [see Littlejohn (1981), and especially Littlejohn (1983) for the general case].

Combining now (2.16) with the Liouville equation (2.8), we easily derive

$$\partial_t|J^{\alpha}|\hat{f}'^{\alpha}=\frac{\partial}{\partial \boldsymbol{Q}}\cdot\left([H_0^{\alpha}, \boldsymbol{Q}]|J^{\alpha}|\hat{f}'^{\alpha}\right)+\frac{\partial}{\partial \boldsymbol{P}}\cdot\left([H_0^{\alpha}, \boldsymbol{P}]|J^{\alpha}|\hat{f}'^{\alpha}\right). \tag{2.17}$$

This is the basic equation of evolution for the *true distribution function* $|J^{\alpha}|\hat{f}'^{\alpha}$. It will be called *the second form of the Liouville equation*. We may

immediately point out a feature that will prove useful in applications: this equation has the explicit form of a *conservation equation in phase space*. The time-derivative of the density $|J^\alpha|\hat{f}'^\alpha$ equals the divergence of a flux in phase space.

We now illustrate these general results on some examples. The first one is quite important. We have already stressed in section **1.4** the fact that the canonical momentum $\boldsymbol{p}$ is a very inconvenient and unphysical variable. A description in terms of the *velocity* $\boldsymbol{v}$ is much to be preferred. We thus perform the pseudo-canonical transformation described in table **1.4.2**. Under this transformation, the distribution function transforms as (see eq. 2.12)

$$\hat{f}^\alpha(\boldsymbol{q}, \boldsymbol{p}; t) \to m_\alpha^3 \hat{f}'^\alpha(\boldsymbol{q}, \boldsymbol{v}; t).$$

We note that, though the transformation is not canonical, the Jacobian J^α is particularly simple, being a *constant* (independent of $\boldsymbol{q}$ and $\boldsymbol{v}$). We take advantage of this in the following way.

In all the previous discussions, the distribution function $\hat{f}^\alpha(\boldsymbol{q}, \boldsymbol{p}; t)$ served as a reference, being expressed in terms of canonical variables, and being normalized to N_α. Having constructed the formalism on that basis, we are no longer in need of a "canonical reference frame"; it is much preferable to have a clear physical basis serving this purpose. We therefore define a new "basic" distribution function, by incorporating the Jacobian of the transformation $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{q}, \boldsymbol{v})$ into the new function

$$f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) \equiv m_\alpha^3 \hat{f}'^\alpha(\boldsymbol{q}, \boldsymbol{v}; t). \tag{2.18}$$

(This new, and definitive reference distribution function wears no "hat" any more!). By this definition it follows from (2.13) that the function $f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)$ is normalized in the variables $\boldsymbol{q}$, $\boldsymbol{v}$ as

$$\int \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) = N_\alpha. \tag{2.19}$$

From here on, all the pseudo-canonical changes of variables we may want to do will be referred to the set $(\boldsymbol{q}, \boldsymbol{v})$: in particular, the Jacobian J^α will always be understood as the Jacobian of the transformation $(\boldsymbol{q}, \boldsymbol{v}) \to (\boldsymbol{Q}, \boldsymbol{P})$. (This convention is also used in the tables of chapter 1.)

The basic distribution function obeys the Liouville equation (in its first form):

$$\partial_t f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) = L^\alpha f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t), \tag{2.20}$$

with the Liouvillian, expressed in the particle variables $(\boldsymbol{q}, \boldsymbol{v})$,

$$L^\alpha = [H_0^\alpha, \boldsymbol{q}] \cdot \frac{\partial}{\partial \boldsymbol{q}} + [H_0^\alpha, \boldsymbol{v}] \cdot \frac{\partial}{\partial \boldsymbol{v}},$$

or, explicitly using table 1.4.2,

$$L^\alpha = -\boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{q}} - \frac{e_\alpha}{m_\alpha}\left(\frac{1}{c}[\boldsymbol{v} \wedge \boldsymbol{B}(\boldsymbol{q})] + \boldsymbol{E}_0(\boldsymbol{q})\right) \cdot \frac{\partial}{\partial \boldsymbol{v}}, \tag{2.21}$$

with

$$\boldsymbol{E}_0(\boldsymbol{q}) = -\frac{\partial}{\partial \boldsymbol{q}} \Phi_0(\boldsymbol{q}). \tag{2.22}$$

The second form of the Liouville equation is almost trivially derived in this case, where $J^\alpha = 1$, by simply noting that $(\boldsymbol{v} \wedge \boldsymbol{B})$ commutes with $\partial/\partial \boldsymbol{v}$:

$$\partial_t f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) = \mathscr{L}^\alpha f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t), \tag{2.23}$$

with

$$\mathscr{L}^\alpha = -\frac{\partial}{\partial \boldsymbol{q}} \cdot \boldsymbol{v} - \frac{\partial}{\partial \boldsymbol{v}} \cdot \frac{e_\alpha}{m_\alpha}\left(\frac{1}{c}[\boldsymbol{v} \wedge \boldsymbol{B}(\boldsymbol{q})] + \boldsymbol{E}_0(\boldsymbol{q})\right). \tag{2.24}$$

Clearly, $\mathscr{L}^\alpha$ must be considered as an *operator*; in other words, the derivatives $\partial/\partial \boldsymbol{q}$, $\partial/\partial \boldsymbol{v}$ act on everything that is written to their right [including, for instance, f^α in eq. (2.23)].

From here on, we may use the formalism of pseudo-canonical transformations for performing very simply any required change of variables. As a particular illustration, which will prove very important in the study of transport in magnetically confined plasmas, we consider the transformation to any of the *natural guiding centre (NGC) variables* discussed in section 1.8, for instance, the set $(\boldsymbol{Y}, \mathscr{E}, M, \phi)$, table 1.8.3. The enormous advantage of the pseudo-canonical formalism lies in the fact that the transformation of the Liouville equation does not require the extremely tedious work of transforming the derivatives $\partial/\partial \boldsymbol{q}$, $\partial/\partial \boldsymbol{v}$ by the chain rule [a mere look at eqs. (1.8.2)–(1.8.5) would discourage many people]; rather, the Lie brackets in eq. (2.8) are directly translated into the appropriate, simple expressions in the new variables, taken from table 1.8.3. Thus, we call $f^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t)$ the distribution function in the new variables *.

* We shall henceforth omit the primes used, e.g. in eq. (2.10). One should never forget, however, the meaning of this abridged notation: $f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) \to f^\alpha(\boldsymbol{q}(\boldsymbol{Y}, \mathscr{E}, M, \phi), \boldsymbol{v}(\boldsymbol{Y}, \mathscr{E}, M, \phi); t) = f'^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t) \equiv f^\alpha(\boldsymbol{Y}, \mathscr{E}, m, \phi; t)$.

The first form of the Liouville equation is

$$\partial_t f^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t) = L^\alpha f^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t), \tag{2.25}$$

with:

$$L^\alpha = [H_0^\alpha, \boldsymbol{Y}] \cdot \frac{\partial}{\partial \boldsymbol{Y}} + [H_0^\alpha, \mathscr{E}] \frac{\partial}{\partial \mathscr{E}} + [H_0^\alpha, M] \frac{\partial}{\partial M} + [H_0^\alpha, \phi] \frac{\partial}{\partial \phi}$$

$$= -U_\alpha \boldsymbol{b}_\alpha^{**} \cdot \frac{\partial}{\partial \boldsymbol{Y}} - \left(\epsilon^{-1}\Omega_\alpha + U_\alpha \boldsymbol{b} \cdot \boldsymbol{R} - \tfrac{1}{2} U_\alpha \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})\right) \frac{\partial}{\partial \phi}. \tag{2.26}$$

Clearly, the quantities U_α, Ω_α, $\boldsymbol{b}_\alpha^{**}$ must be provided with a species index α, because they must be evaluated with different parameters for ions and for electrons. Thus

$$U_\alpha = \sigma \{(2/m_\alpha)[\mathscr{E} - e_\alpha \Phi_0(\boldsymbol{Y}) - MB(\boldsymbol{Y})]\}^{1/2}, \tag{2.27}$$

$$\Omega_\alpha = \frac{e_\alpha B(\boldsymbol{Y})}{m_\alpha c}, \tag{2.28}$$

and an obvious definition for $\boldsymbol{b}_\alpha^{**}$. *Note that*

$$\Omega_e < 0, \qquad \Omega_i > 0.$$

The second form of the Liouville equation is

$$\partial_t \left(\frac{B_{\|\alpha}^*}{m_\alpha^2 |U_\alpha|} f^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t) \right) = \mathscr{L}^\alpha \left(\frac{B_{\|\alpha}^*}{m_\alpha^2 |U_\alpha|} f^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t) \right), \tag{2.29}$$

where we used the Jacobian taken from table 1.8.3; in the present case it is no longer a constant, but a non-trivial function of $\boldsymbol{Y}$, $\mathscr{E}$, M,

$$|J^\alpha| = \frac{B_{\|\alpha}^*}{m_\alpha^2 |U_\alpha|} = \frac{B(\boldsymbol{Y})}{m_\alpha^2 |U_\alpha|} \left(1 + \frac{\epsilon}{\Omega_\alpha(\boldsymbol{Y})} U_\alpha \boldsymbol{b}(\boldsymbol{Y}) \cdot [\nabla \wedge \boldsymbol{b}(\boldsymbol{Y})]\right). \tag{2.30}$$

The operator $\mathscr{L}^\alpha$ is the "transpose" of the operator L^α:

$$\mathscr{L}^\alpha = -\frac{\partial}{\partial \boldsymbol{Y}} \cdot \dot{\boldsymbol{Y}} - \frac{\partial}{\partial \mathscr{E}} \dot{\mathscr{E}} - \frac{\partial}{\partial M} \dot{M} - \frac{\partial}{\partial \phi} \dot{\phi}. \tag{2.31}$$

This example is sufficient for illustrating the use of the pseudo-canonical transformations. It avoids all tedious calculation of derivatives, or even more tedious averaging operations.

Before concluding this section, we want to settle some questions of vocabulary. Some authors loosely call (2.25)–(2.26) the "Vlasov equation", or even the "collisionless drift kinetic equation". Such names are due to an analogy in form; they are misleading, because the physical content of these various equations is quite different. Equation (2.25) is the (reduced) *Liouville equation*, or equivalently, the first equation of the BBGKY hierarchy, for a system of non-interacting particles in the presence of an external electromagnetic field. The Vlasov equation, on the contrary, describes a system of interacting charged particles: it will be derived in section 2.5. As for the drift kinetic equation, it is farther away from the Liouville equation, as it involves a peculiar averaging over the gyrophase, which will be discussed in chapter 10.

2.3. *Liouville equation for independent particles in time-dependent external fields*

We now discuss the extension of the previous results to the case where the external fields derive from time-dependent potentials $\Phi(\boldsymbol{x}, t)$, $\boldsymbol{A}(\boldsymbol{x}, t)$. The main result of section 1.9 is that the Hamiltonian formalism is applicable exactly as in the stationary problem, provided that

- the phase space is extended by adding a pair of canonical variables t, h;
- the Hamiltonian $H(q, p)$ is extended into

$$\tilde{H}(q, p, t, h) \equiv H(q, p, t) + h; \tag{3.1}$$

- the motion is parametrized by a "proper time" τ.

The formulation of statistical mechanics in this framework is very easy in its first steps. All we need to do is define an *extended phase-space distribution function* $\tilde{F}$,

$$\tilde{F}(q, p, h, t; \tau) \equiv \tilde{F}(\boldsymbol{q}_1, \cdots, \boldsymbol{q}_N, \boldsymbol{p}_1, \cdots, \boldsymbol{p}_N, h, t; \tau). \tag{3.2}$$

As h and t are just additional canonical variables, the derivation of an *extended Liouville equation*, generalizing (1.9) and (1.10), is straightforward:

$$\partial_\tau \tilde{F} = \tilde{L}\tilde{F}, \tag{3.3}$$

with

$$\tilde{L} = [\tilde{H}, \cdots],$$
$$= \sum_{j=1}^{N} \left([\tilde{H}, \boldsymbol{q}_j] \cdot \frac{\partial}{\partial \boldsymbol{q}_j} + [\tilde{H}, \boldsymbol{p}_j] \cdot \frac{\partial}{\partial \boldsymbol{p}_j} \right) + [\tilde{H}, h] \frac{\partial}{\partial h} + [\tilde{H}, t] \frac{\partial}{\partial t}. \tag{3.4}$$

Going through the same steps as in section 2.2, we may derive a Liouville equation (in the second form) for the reduced (extended) one-particle distribution function $\tilde{f}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t; \tau)$, expressed in terms of arbitrary, not necessarily canonical variables $\boldsymbol{Q}$, $\boldsymbol{P}$, k, t (see eq. 2.17),

$$\partial_{\tau} | \tilde{f}^{\alpha} | \tilde{f}^{\alpha} = \tilde{\mathscr{L}}^{\alpha} | \tilde{f}^{\alpha} | \tilde{f}^{\alpha}, \tag{3.5}$$

with

$$\tilde{\mathscr{L}}^{\alpha} = \frac{\partial}{\partial \boldsymbol{Q}} \cdot [\tilde{H}_0^{\alpha}, \boldsymbol{Q}] + \frac{\partial}{\partial \boldsymbol{P}} \cdot [\tilde{H}_0^{\alpha}, \boldsymbol{P}] + \frac{\partial}{\partial k} [\tilde{H}_0^{\alpha}, k] + \frac{\partial}{\partial t} [\tilde{H}_0^{\alpha}, t], \tag{3.6}$$

where

$$\tilde{H}_0^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t) = H_0^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, t) + k.$$

All these results are quite straightforward, but purely formal. We are left, however, with the problem of understanding the physical content of the extended phase-space distribution functions and with the interpretation (or elimination!) of the artificially added degrees of freedom. In this context, we meet right away with a difficulty. The statistical interpretation of the distribution function $\tilde{f}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t; \tau)$ requires its normalization over the entire *extended* phase space,

$$\int \mathrm{d}\boldsymbol{Q}\, \mathrm{d}\boldsymbol{P} \int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}t\, | \tilde{f}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t) | \tilde{f}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t; \tau) = N_{\alpha}. \tag{3.7}$$

This is a very strange condition, as it seems to imply that the distribution function is normalized only when integrated over t from $-\infty$ to $+\infty$, i.e. over the whole past and future history of the system!

The missing point is that the set of variables $(\boldsymbol{Q}, P, k, t; \tau)$ which are treated *mathematically* as independent variables are not *physically* independent. At this stage, having completed the Hamiltonian machinery, we are able to take into account the relations expressing the interdependence of these variables and go over to a set of concepts involving only physically independent variables. This operation must be done carefully, in order to avoid inconsistencies. We first prove a few useful lemmas.

Lemma 1. *Consider a set of (generally non-canonical) phase-space variables* $\tilde{Z}^\lambda \equiv (\boldsymbol{Q}, \boldsymbol{P}, k, t)$ *obtained from the basic particle variables* $(\boldsymbol{q}, \boldsymbol{v}, h, t)$ *by a pseudo-canonical transformation for which*

$$[t, \boldsymbol{Q}] = 0, \qquad [t, \boldsymbol{P}] = 0, \qquad [t, k] = 1. \tag{3.8}$$

Then, the extended Jacobian $\tilde{J}^\alpha$ *equals the ordinary Jacobian* J^α *of the transformation* $(\boldsymbol{q}, \boldsymbol{v}) \to (\boldsymbol{Q}, \boldsymbol{P})$.

Proof. Let us use notations similar to (**1**.2.29) and define
- the 8×8 matrix $\tilde{\Sigma}^{\lambda\mu}$,

$$\tilde{\Sigma}^{\lambda\mu} = [\tilde{Z}^\lambda, \tilde{Z}^\mu], \qquad \tilde{Z}^\lambda = (\boldsymbol{Q}, \boldsymbol{P}, k, t);$$

- the 6×6 matrix Σ^{lm},

$$\Sigma^{lm} = [Z^l, Z^m], \qquad Z^l = (\boldsymbol{Q}, \boldsymbol{P}).$$

With conditions (3.8), the matrix $\tilde{\Sigma}$ has the form

$$\tilde{\Sigma} = \begin{pmatrix} & & & & & & * & 0 \\ & & & & & & * & 0 \\ & & & \Sigma^{lm}\ (6\times 6) & & & * & 0 \\ & & & & & & * & 0 \\ & & & & & & * & 0 \\ & & & & & & * & 0 \\ * & * & * & * & * & * & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

where $*$ denotes some matrix elements, not necessarily null. Clearly, because of the peculiar outermost row and column, the determinant of this matrix equals the determinant of the matrix Σ:

$$\| \tilde{\Sigma}^{\lambda\mu} \| = (-1) \cdot (-1) \cdot \| \Sigma^{lm} \|.$$

The lemma then follows from Littlejohn's theorem (**1**.2.36).

Lemma 2. *If the fundamental Lie brackets of the variables $\tilde{Z}^\lambda$ are independent of the variable k, the Jacobian is also independent of k.*

This property follows trivially from (1.2.36).

Lemma 3. *If the conditions of Lemma 1 are fulfilled, the equation of motion of t is*

$$\dot{t} = 1. \tag{3.9}$$

Lemma 4. *The following relation holds quite generally:*

$$\dot{H}_0^\alpha(\boldsymbol{Q}, \boldsymbol{P}, t) = -\dot{k}. \tag{3.10}$$

Indeed, from (3.1) we get

$$\dot{H}_0^\alpha = \left[H_0^\alpha, \tilde{H}_0^\alpha\right] = \left[H_0^\alpha, H_0^\alpha\right] + \left[H_0^\alpha, k\right] = \left[H_0^\alpha, k\right],$$

$$\dot{k} = \left[k, \tilde{H}_0^\alpha\right] = \left[k, H_0^\alpha\right] + \left[k, k\right] = \left[k, H_0^\alpha\right].$$

From lemmas 3 and 4 we might be tempted to conclude (disregarding irrelevant integration constants) that

$$t = \tau, \tag{3.11}$$

$$H_0^\alpha(\boldsymbol{Q}, \boldsymbol{P}, t) = -k, \tag{3.12}$$

or, equivalently,

$$\tilde{H}_0^\alpha(\boldsymbol{Q}, \boldsymbol{P}, k, t) = 0. \tag{3.12a}$$

These relations may be used for the elimination of the subsidiary variables t and k.

However, one must be very careful in handling such equalities. These are typical examples of the class of relations called *"weak equalities"* or *"subsidiary conditions"*. Their peculiar nature was first discovered by Dirac (1949), then further developed by Dirac (1985) (this concept was extensively used in his work on the Hamiltonian formulation of gravitation); a quite extensive account is found in the book by Sudarshan and Mukunda (1974). We quote here a paragraph from the article by Dirac (1958), which clearly explains the problem: *"The definition* (1.2.18) *of the Poisson bracket (P.b.) requires the q's and the p's to be considered as independent variables. Any relations which restrict*

this independence of the q's and p's [such as eqs. (3.11), (3.12)] *must not be used before one works out P.b.'s or the P.b.'s would cease to be well-defined quantities. To remind us of this limitation in the use of some of our equations, it is convenient to call such equations weak equations...*". [The equation numbers are those of the present work.]

Whereas the correct handling of weak equations is always somewhat delicate in the framework of ordinary dynamics, the distinction between strong and weak equations can be made very clear and simple in the framework of statistical mechanics. All the operations leading to (3.5) involve only strong equations, i.e. *Lie brackets*, in which k and t are treated as independent variables. At this stage we make use of the subsidiary conditions by requiring the following principle: *The only physically accessible region of the extended phase space is the region where the weak equations* (3.11), (3.12) *are satisfied.*

This implies that the distribution function $\tilde{f}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t; \tau)$ for a physical system must obey the following normalization condition, which replaces (3.7):

$$\int \mathrm{d}\boldsymbol{Q}\,\mathrm{d}\boldsymbol{P} \int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}t\, |\tilde{J}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t)|\, \delta\big(\tilde{H}_0^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t)\big)$$

$$\times \delta(t-\tau)\tilde{f}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t; \tau) = N_{\alpha}. \tag{3.13}$$

Thus, we arrive at the following general rule: *the weak equations must be expressed by δ-functions which restrict the extended phase space to the physically accessible region.* In this way, the unacceptable features of postulate (3.7) are suppressed.

At this stage, the unwanted variables can be explicitly eliminated. We first perform the trivial integration over t,

$$\int \mathrm{d}\boldsymbol{Q}\,\mathrm{d}\boldsymbol{P}\, |J^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, \tau)| \int_{-\infty}^{\infty} \mathrm{d}k\, \delta\big(\tilde{H}_0^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, \tau)\big)\tilde{f}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, \tau; \tau)$$

$$= N_{\alpha}. \tag{3.14}$$

In writing this equation, we made explicit use of lemmas 1 and 2. We now introduce the *physical distribution function* $f^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}; t)$ by the following definition:

$$f^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}; t) \equiv \int_{-\infty}^{\infty} \mathrm{d}k\, \delta(H_0^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, t) + k)\tilde{f}^{\alpha}(\boldsymbol{Q}, \boldsymbol{P}, k, t; t). \tag{3.15}$$

As a result of lemmas 1 and 2, this distribution function obeys the "usual" normalization (2.13)

$$\int \mathrm{d}\boldsymbol{Q}\,\mathrm{d}\boldsymbol{P}\,|\,J^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,t)\,|\,f^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P};\,t) = N_{\alpha}. \tag{3.16}$$

Finally, we derive the *physical Liouville equation* (in the second form) by a straightforward procedure from (3.5) and (3.6):

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\int \mathrm{d}k\,\mathrm{d}t\,\delta(H_0^{\alpha}+k)\delta(t-\tau)\,|\,J^{\alpha}\,|\,\tilde{f}^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,k,\,t;\,\tau)\\
&\quad\equiv \frac{\partial}{\partial\tau}\,|\,J^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,\tau)\,|\,f^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P};\,\tau)\\
&\quad= \int \mathrm{d}k\,\mathrm{d}t\,\delta(H_0^{\alpha}+k)\delta(t-\tau)\\
&\qquad\times\left(\frac{\partial}{\partial\boldsymbol{Q}}\cdot[H_0^{\alpha},\,\boldsymbol{Q}] + \frac{\partial}{\partial\boldsymbol{P}}\cdot[H_0^{\alpha},\,\boldsymbol{P}] + \frac{\partial}{\partial k}[H_0^{\alpha},\,k] + \frac{\partial}{\partial t}[k,\,t]\right)\\
&\qquad\times\,|\,J^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,t)\,|\,\tilde{f}^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,k,\,t;\,\tau)\\
&\qquad+\int \mathrm{d}k\,\mathrm{d}t\left(\frac{\partial}{\partial\tau}\delta(t-\tau)\delta(H_0^{\alpha}+k)\right)\\
&\qquad\quad\times\,|\,J^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,t)\,|\,\tilde{f}^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,k,\,t;\,\tau).
\end{aligned}$$

A careful calculation, involving a partial integration over t, the use of Lemma 4 and a final renaming of $\tau \to t$, leads to the result

$$\begin{aligned}
&\frac{\partial}{\partial t}\,|\,J^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,t)\,|\,f^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P};\,t)\\
&\quad=\left(\frac{\partial}{\partial\boldsymbol{Q}}\cdot[H_0^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,t),\,\boldsymbol{Q}] + \frac{\partial}{\partial\boldsymbol{P}}\cdot[H_0^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,t),\,\boldsymbol{P}]\right)\\
&\qquad\times\,|\,J^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P},\,t)\,|\,f^{\alpha}(\boldsymbol{Q},\,\boldsymbol{P};\,t),
\end{aligned} \tag{3.17}$$

or, written more compactly,

$$\partial_t\,|\,J^{\alpha}(t)\,|\,f^{\alpha}(t) = \mathscr{L}^{\alpha}(t)\,|\,J^{\alpha}(t)\,|\,f^{\alpha}(t). \tag{3.18}$$

Finally, eq. (2.16) is easily extended to the non-stationary case; this leads to the derivation of the Liouville equation in its first form,

$$\partial_t f^\alpha(t) = L^\alpha(t) f^\alpha(t), \tag{3.19}$$

with

$$L^\alpha(t) = \left[H_0^\alpha(\boldsymbol{Q}, \boldsymbol{P}, t), \boldsymbol{Q}\right] \cdot \frac{\partial}{\partial \boldsymbol{Q}} + \left[H_0^\alpha(\boldsymbol{Q}, \boldsymbol{P}, t), \boldsymbol{P}\right] \cdot \frac{\partial}{\partial \boldsymbol{P}}. \tag{3.20}$$

Our results can be summarized as follows. The distribution function $f^\alpha(\boldsymbol{Q}, \boldsymbol{P}; t)$ satisfies, in the non-stationary case, a Liouville equation which is formally constructed from the Hamiltonian by the same procedure as in the stationary case. In other words, the time variable in the Hamiltonian – and thus in the Liouvillian – appears as a mere parameter. The extreme simplicity of this final result should not hide its non-trivial nature. We also stress that the results obtained here do not depend in any way on assumptions about the rapidity of the time-dependence of the external fields; they are valid equally well for fast and for slow time variations.

As an illustration of these results, we write down explicitly the Liouville equation, expressed in terms of the natural guiding centre variables $(\boldsymbol{Y}, \mathscr{E}, M, \phi)$, derived in section **1.9** for a slowly time-dependent external field. In this case:

$$L^\alpha(t) = \dot{\boldsymbol{Y}} \cdot \frac{\partial}{\partial \boldsymbol{Y}} + \dot{\mathscr{E}} \frac{\partial}{\partial \mathscr{E}} + \dot{M} \frac{\partial}{\partial M} + \dot{\phi} \frac{\partial}{\partial \phi}.$$

This equation must be combined with the equations of motion collected in table **1.9.1**. The difference with eqs. (2.25) and (2.26) is that $\dot{\mathscr{E}} \neq 0$ in the non-stationary case. We thus obtain

$$\partial_t f^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t) = L^\alpha(t) f^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t), \tag{3.21}$$

with

$$\begin{aligned} L^\alpha(t) = &- U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}} - \epsilon \boldsymbol{V}_{\mathrm{D}}^\alpha \cdot \frac{\partial}{\partial \boldsymbol{Y}} + \frac{e_\alpha}{c} U_\alpha \left(\boldsymbol{b} \cdot \partial_t \boldsymbol{A}(\boldsymbol{Y}, t)\right) \frac{\partial}{\partial \mathscr{E}} \\ &- \left(\frac{1}{\epsilon} \Omega_\alpha + U_\alpha \boldsymbol{b} \cdot \boldsymbol{R} - \tfrac{1}{2} U_\alpha \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})\right) \frac{\partial}{\partial \phi}, \end{aligned} \tag{3.22}$$

where we introduced the *drift velocity* $\boldsymbol{V}_{\mathrm{D}}^\alpha$ defined as

$$\epsilon \boldsymbol{V}_{\mathrm{D}}^\alpha = \frac{\epsilon}{m_\alpha \Omega_\alpha} \boldsymbol{b} \wedge \left(m_\alpha U_\alpha^2 (\boldsymbol{b} \cdot \nabla) \boldsymbol{b} + M \nabla B + e_\alpha \nabla \Phi\right). \tag{3.23}$$

This equation will play a leading role in the neoclassical theory of transport (chaps. 10–16).

2.4. *The BBGKY equations and the kinetic equation for interacting charged particles*

We now go one step higher in the complexity (and also in the realism) of the description of a plasma, by taking account of the *interactions* among the charged particles. The motion of the latter is determined not only by the external electromagnetic fields (the source of which is outside the plasma), but also by the fields produced by the particles themselves. Each particle in the plasma feels, besides the external field, an electromagnetic field which is wildly varying, as it depends on the instantaneous coordinates and velocities of all the other particles in the system. This is the essence of the interaction effect.

The simplest way of accounting for the interactions in plasmas is by postulating the following form of the Hamiltonian [expressed in the basic particle variables $(\boldsymbol{q}_1, \boldsymbol{v}_1, \cdots, \boldsymbol{q}_N, \boldsymbol{v}_N) \equiv (q, v)$]:

$$H(q, v) = \sum_{j=1}^{N} \left[\tfrac{1}{2} m_{\alpha_j} v_j^2 + e_{\alpha_j} \Phi_0(\boldsymbol{q}_j) \right] + \sum_{j<n=1}^{N} \sum V_{jn}^{\alpha_j \alpha_n}(\boldsymbol{q}_j, \boldsymbol{q}_n)$$

$$\equiv \sum_{j=1}^{N} H_{0j}^{\alpha_j} + \sum_{j<n=1}^{N} \sum V_{jn}^{\alpha_j \alpha_n}. \tag{4.1}$$

The new feature, as compared to (2.2), is the appearance of a term that is a sum, over all distinct *pairs* of particles, of functions depending non-additively on the positions of *two particles*, j and n (of species α_j, α_n). We assume that the *interaction potential* describes the *Coulomb interaction* of two charged point particles:

$$V_{jn}^{\alpha_j \alpha_n} \equiv V^{\alpha_j \alpha_n}(\boldsymbol{q}_j, \boldsymbol{q}_n) = e_{\alpha_j} e_{\alpha_n} \frac{1}{|\boldsymbol{q}_j - \boldsymbol{q}_n|}. \tag{4.2}$$

$V_{jn}^{\alpha_j \alpha_n}$ is thus a function of the relative distance between the two particles, and is independent of their velocities.

It must be realized that the assignment (4.2) has several limitations, of which we discuss the following two.

(1) Equation (4.2) assumes that the plasma is *fully ionized*. In other words, it excludes the presence of neutral particles (see the discussion in section 2.1). We also assume that the charge state of all the ions is permanent, i.e. we do

not consider such processes as ionization, recombination or charge exchange. For all these excluded processes we would need to take into account complicated, non-Coulombic interactions. Moreover, a classical description is inadequate for processes in which the internal structure of the atoms is relevant: quantum mechanics becomes indispensable.

(2) Even for a fully ionized plasma, (4.2) represents a *non-relativistic approximation* of the interaction process. It is not easy to formulate a general validity criterion for this approximation. Roughly speaking, we may say that first-order relativistic corrections (i.e. terms of order v/c) may be important under not too extreme conditions. Indeed, the source term for the magnetic field in Maxwell's equations is $4\pi(\boldsymbol{j}/c)$, and the electric current density $\boldsymbol{j}$ is defined in terms of the average relative velocity of the particles. Thus, a consistent macroscopic description of the plasma requires a dynamical theory incorporating at least the first-order relativistic effects. This point will be discussed again at the end of section 2.5.

On the other hand, the higher-order corrections [of order $(v/c)^2$ or higher] are necessary for a relativistic description of the collision processes. These effects become observable only when there is a sizeable fraction of the particles having velocities close to the speed of light. Alternatively, it may be said that the thermal energy of the particles becomes of the order of their rest-mass energy. Such extreme conditions can occur (marginally) in some laser-fusion experiments, and are typical of the interior of some classes of stars (such as the white dwarfs).

Finally, it may be added that the study of the interactions of magnetically confined plasmas with a radio-frequency field used for their heating and for current-drive experiments may require a relativistic theory, because the relevant (= resonant) particles involved in the interaction are those of the high-energy tail of the distribution function.

The fully relativistic description of interacting systems is by no means a simple problem: it will not be discussed here (see Klimontovich 1964, Balescu and Kotera 1967, Balescu et al. 1967, de Groot et al. 1980).

In spite of these limitations, the model of a *fully ionized non-relativistic plasma* covers a large spectrum of interesting problems and will be the main object of this book. The first stages of the implementation of this description closely follow the beginning of section 2.1. We define again a phase-space distribution function $F(q, v; t)$ (which we directly express in terms of velocity, rather than momentum variables). This function obeys the Liouville equation (1.8) in which, of course, the Hamiltonian is now eq. (4.1). In the next step, we introduce reduced distribution functions and derive from the Liouville equation an equation for the reduced one-particle distribution function $f^{\alpha}(\boldsymbol{q}_1, \boldsymbol{v}_1; t)$ (see Balescu 1975). At this point arises the essential difference introduced by the interactions. The latter equation is *not closed* for

$f^{\alpha}(\boldsymbol{q}_1, \boldsymbol{v}_1; t)$, but contains a term involving the two-particle reduced distribution function $f^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{v}_1, \boldsymbol{q}_2, \boldsymbol{v}_2; t)$,

$$\partial_t f^{\alpha}(\boldsymbol{q}_1, \boldsymbol{v}_1; t)$$

$$= L_1^{\alpha} f^{\alpha}(\boldsymbol{q}_1, \boldsymbol{v}; t) + \sum_{\beta=\mathrm{e,i}} \int \mathrm{d}\boldsymbol{q}_2 \, \mathrm{d}\boldsymbol{v}_2 \; L_{12}'^{\alpha\beta} \, f^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{v}_1, \boldsymbol{q}_2, \boldsymbol{v}_2; t). \quad (4.3)$$

The operator $L_{12}'^{\alpha\beta}$ is the *interaction Liouvillian*, describing the interaction of particles 1 (of species α) and 2 (of species β):

$$L_{12}'^{\alpha\beta} \, G = \left[V_{12}^{\alpha\beta}, G \right] \quad (4.4)$$

for any function G of the coordinates of particles 1 and 2.

The non-closed character of eq. (4.3) reflects the basic difficulty of statistical mechanics: if we derive an equation for $f^{\alpha\beta}$, it will contain the three-particle distribution function, and so on. Thus, the exact solution of the problem requires the solution of N (practically: ∞) coupled equations for all the reduced distribution functions: the well-known *BBGKY hierarchy*.

This is not the place for a detailed discussion of the various approximation methods used in statistical mechanics and of their justification (see Balescu 1975). We shall use in this book a simple form of the *weak coupling approximation*, which is sufficient for treating many problems of interest. In order to define this approximation, we first introduce the concept of a binary *correlation function* $g^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{v}_1, \boldsymbol{q}_2, \boldsymbol{v}_2; t)$,

$$g^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{v}_1, \boldsymbol{q}_2, \boldsymbol{v}_2; t)$$

$$= f^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{v}_1, \boldsymbol{q}_2, \boldsymbol{v}_2; t) - f^{\alpha}(\boldsymbol{q}_1, \boldsymbol{v}_1; t) f^{\beta}(\boldsymbol{q}_2, \boldsymbol{v}_2; t). \quad (4.5)$$

The correlation function measures the deviation from statistical independence of the two particles. It is expected, on physical grounds, that $g^{\alpha\beta} \to 0$ as the relative distance between the particles increases. One may usually define a quantity r_c, called the *range of the correlations*, noting that

$$g^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{v}_1, \boldsymbol{q}_2, \boldsymbol{v}_2; t) \approx 0 \quad \text{for} \quad |\boldsymbol{q}_1 - \boldsymbol{q}_2| > r_c.$$

The following normalization property follows from eq. (4.5), and eqs. (1.14) and (1.15):

$$\int \mathrm{d}\boldsymbol{q}_1 \, \mathrm{d}\boldsymbol{v}_1 \, \mathrm{d}\boldsymbol{q}_2 \, \mathrm{d}\boldsymbol{v}_2 \; g^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{v}_1, \boldsymbol{q}_2, \boldsymbol{v}_2; t) = 0. \quad (4.6)$$

One can similarly define three-body, four-body,... correlation functions. The physical origin of the correlations is, naturally, the existence of interactions, which cause the particles to feel their mutual influence.

It is well known that, in a plasma, the relative importance of the Coulomb interactions, compared to the mean kinetic energy (i.e. the thermal energy) is measured by the characteristic dimensionless parameter μ_P, involving the charge e_α, the mean particle density n_α and the temperature T_α:

$$\mu_P = \left(\tfrac{4}{3}\pi\lambda_D^3 n_i\right)^{-2/3} = (36\pi)^{1/3}\frac{e^2 n_i^{-1/3}(ZT_e + T_i)}{T_e T_i(1+Z)}, \tag{4.7}$$

where λ_D is the *Debye length* (Trubnikov 1965),

$$\lambda_D = \left(\frac{4\pi Z e^2(n_e T_e + n_i T_i)}{T_e T_i(1+Z)}\right)^{-1/2}. \tag{4.8}$$

The plasma parameter is defined by various authors in slightly different ways, differing by a numerical factor, or by the exponent in eq. (4.7). The definition adopted here has a very simple physical interpretation. Up to the factor $(36\ \pi)^{1/3} \simeq 4.836$, the right-hand side of eq. (4.7) represents the ratio of the Coulomb energy, evaluated at the mean distance of the ions, $n_i^{-1/3}$, to the mean thermal energy, defined as

$$T^{-1} = \frac{ZT_e + T_i}{(1+Z)T_e T_i} = \left(\frac{1}{n_i T_i} + \frac{1}{n_e T_e}\right)\Big/\left(\frac{1}{n_i} + \frac{1}{n_e}\right),$$

where we used the electroneutrality condition $n_e = Zn_i$ and where the temperature is expressed in energy units (see section 3.2). Other authors prefer to use the parameter N_D,

$$N_D = \tfrac{4}{3}\pi\lambda_D^3 n_i = \mu_P^{-3/2}. \tag{4.9}$$

The parameter N_D represents the number of particles in a sphere of radius equal to the Debye length λ_D.

We define a plasma as being *weakly coupled* when the following relation is satisfied:

$$\mu_P \ll 1. \tag{4.10}$$

(This implies $N_D \gg 1$.) Clearly, this condition can be met in two ways. For a given temperature, the condition is satisfied for sufficiently *low density*; alternatively, for a given density, it holds for sufficiently *high temperatures*.

Thus, in a weak coupling regime, the potential interaction energy of the particles is, on the average, very small compared to their mean kinetic energy. From the previous discussion, it follows that the correlation function $g^{\alpha\beta}$ is also small under such circumstances, typically of the same order as the interaction potential. As for the s-body correlation functions ($s > 2$), they are of higher order of smallness. A natural truncation of the BBGKY hierarchy is obtained by neglecting all correlation functions of more than two particles. The hierarchy then reduces to a set of two coupled equations.

The equation for f^{α} is obtained by combining (4.3) with (4.5) [and using the obvious abbreviation j for the set $(\boldsymbol{q}_j, \boldsymbol{v}_j)$],

$$\partial_t f^{\alpha}(1;\, t) = L_1^{\alpha} f^{\alpha}(1;\, t) + \sum_{\beta} \int \mathrm{d}2\; L_{12}'^{\alpha\beta}\; f^{\alpha}(1;\, t) f^{\beta}(2;\, t)$$

$$+ \sum_{\beta} \int \mathrm{d}2\; L_{12}'^{\alpha\beta}\; g^{\alpha\beta}(1, 2;\, t). \tag{4.11}$$

This equation is coupled to the equation for the correlation function (see Balescu 1975), in which the three-body correlation is set equal to zero:

$$\begin{aligned}
&\partial_t g^{\alpha\beta}(1, 2;\, t) \\
&\quad = \left(L_1^{\alpha} + L_2^{\beta}\right) g^{\alpha\beta}(1, 2;\, t) + L_{12}'^{\alpha\beta}\; g^{\alpha\beta}(1, 2;\, t) \\
&\qquad + \sum_{\gamma} \int \mathrm{d}3 \Big[L_{13}'^{\alpha\gamma}\; f^{\alpha}(1;\, t) g^{\beta\gamma}(2, 3;\, t) + L_{23}'^{\beta\gamma}\; f^{\beta}(2;\, t) g^{\alpha\gamma}(1, 3;\, t) \\
&\qquad\quad + \left(L_{13}'^{\alpha\gamma} + L_{23}'^{\beta\gamma}\right) f^{\gamma}(3;\, t) g^{\alpha\beta}(1, 2;\, t) \Big] + L_{12}'^{\alpha\beta} f^{\alpha}(1;\, t) f^{\beta}(2;\, t)
\end{aligned} \tag{4.12}$$

Even this set of two coupled equations for f^{α} and $g^{\alpha\beta}$ is too complicated for explicit calculations. By using arguments which will not be described in detail for lack of space, it can be shown that the following two simpler versions of (4.12) are appropriate for describing a plasma. In a first stage, we retain

$$\begin{aligned}
&\partial_t g^{\alpha\beta}(1, 2;\, t) - \left(L_1^{\alpha} + L_2^{\beta}\right) g^{\alpha\beta}(1, 2;\, t) \\
&\quad = \sum_{\gamma} \int \mathrm{d}3 \left[L_{13}'^{\alpha\gamma} f^{\alpha}(1;\, t) g^{\beta\gamma}(2, 3;\, t) + L_{23}'^{\beta\gamma} f^{\beta}(2;\, t) g^{\alpha\gamma}(1, 3;\, t) \right] \\
&\qquad + L_{12}'^{\alpha\beta}\; f^{\alpha}(1;\, t) f^{\beta}(2;\, t). \qquad [BL]
\end{aligned} \tag{4.13}$$

The second, simpler version is

$$\partial_t g^{\alpha\beta}(1, 2; t) - \left(L_1^\alpha + L_2^\beta\right) g^{\alpha\beta}(1, 2; t)$$
$$= L_{12}'^{\alpha\beta} f^\alpha(1; t) f^\beta(2; t). \qquad [L] \tag{4.14}$$

The main difference between the two approximations can be readily recognized. In eq. (4.14), only the two particles 1 and 2, appearing in the unknown correlation function, are involved. One therefore expects this approximation to describe an evolution driven by *binary collisions* of the particles (in the weak coupling approximation). The end result of this approximation will be the *Landau kinetic equation*. On the contrary, in eq. (4.13), we also retain interactions with a third particle taken from the "bath". This implies that we admit a certain class of many-body interactions in the approximate description of the evolution. The product of this procedure is the *Balescu–Lenard equation*, which contains a dynamic description of the screening or polarization) in the plasma, a typical collective effect.

The last stage in reaching a true kinetic equation is the elimination of the correlation function between eqs. (4.11) and (4.13) or (4.14). This is a crucial and non-trivial step, which will be discussed in section 2.6. Let us say here that if the formal solution of (4.14) or (4.13) is substituted into (4.11), the resulting equation is quite untractable, because of its mathematical complexity. A careful study reveals, however, the possible emergence of a regime in which the correlation function at time t becomes a functional of the one-particle distribution functions at the same time t (Bogoliubov 1962, Balescu 1975):

$$g^{\alpha\beta}(t) = g^{\alpha\beta}\{f(t)\}.$$

When this *kinetic regime* is established, the substitution of this correlation function into (4.11) results in

$$\partial_t f^\alpha(1; t) = L_1^\alpha f^\alpha(1; t) + \sum_\beta \int \mathrm{d}2\, L_{12}'^{\alpha\beta} f^\alpha(1; t) f^\beta(2; t) + \mathscr{K}^\alpha\{f(t)\}. \tag{4.15}$$

This is the typical, generic form of a *kinetic equation*. This concept is defined as a *closed, Markovian, generally non-linear equation of evolution for the one-particle distribution function* $f^\alpha(1; t)$. The word "Markovian" means (broadly speaking) that the kinetic equation only involves the value of the unknown distribution function at time t. In particular, the rate of change $\partial_t f^\alpha(1; t)$ is not influenced by the values taken by the distribution function in the past, i.e. $f^\alpha(1; t-\tau)$, $\tau > 0$.

We may already characterize the role of the various terms of (4.15) in driving the change in time of the distribution function. L_1^α describes the effect of the free motion (kinetic energy) as well as the effect of the external electric and magnetic fields. The interactions affect the motion in two ways. The second term on the right-hand side will be shown to represent the effect of an average self-consistent field acting on particle 1 and produced by all the other particles in the system. Besides this "smooth" background action, the interactions also act more "violently" upon a sufficiently close approach of two (or more) particles. This important effect is contained in the *collision term* $\mathscr{K}^\alpha$. Its specific form depends widely on the nature and the state of the system. For instance, it is quite different when only binary collisions are considered or when collective effects are included; its form depends on the density and the temperature, being quite different for a fusion plasma or for a degenerate quantum plasma; it can be affected by the presence of a (very strong) electric or magnetic field, etc.

The most important property of (4.15) is the following. When the kinetic regime is established, *the Hamiltonian (or Liouvillian) structure of the evolution process is lost*. The detailed reason for this "symmetry breaking" will appear in section 2.6. Let us say here that this feature opens the gate towards the description of a *dissipative and irreversible evolution*. This property was absent in the initial Hamiltonian description, but is indispensable for understanding macroscopic physics.

2.5. *The Vlasov kinetic equation*

We now derive the explicit form of the kinetic equation for the one-particle distribution function $f^\alpha(1;\, t)$ in the simplest non-trivial case. We therefore go back to eq. (4.15) and analyze its various contributions. We first note that the interaction Liouvillian $L_{12}^{\prime\alpha\beta}$ has the form

$$L_{12}^{\prime\alpha\beta} = \left[\nabla_1 V^{\alpha\beta}(\boldsymbol{q}_1 - \boldsymbol{q}_2)\right] \cdot (\partial_1 - \partial_2)$$

$$= e_\alpha e_\beta \left(\nabla_1 \frac{1}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|} \right) \cdot (\partial_1 - \partial_2), \tag{5.1}$$

where the following abbreviations were used:

$$\nabla_1 \equiv \frac{\partial}{\partial \boldsymbol{q}_1}, \qquad \partial_1 \equiv \frac{\partial}{\partial \boldsymbol{v}_1}. \tag{5.2}$$

The second term on the right-hand side of (4.15) can be transformed as

$$\sum_{\beta} \int \mathrm{d}2\; L_{12}'^{\alpha\beta}\; f^{\alpha}(1;\, t) f^{\beta}(2;\, t)$$

$$= \sum_{\beta} \int \mathrm{d}2\; \left(\nabla_1 V_{12}^{\alpha\beta}\right) \cdot \left(m_{\alpha}^{-1} \partial_1 - m_{\beta}^{-1} \partial_2\right) f^{\alpha}(1;\, t) f^{\beta}(2;\, t)$$

$$= \left(\nabla_1 \sum_{\beta} \int \mathrm{d}2\; V_{12}^{\alpha\beta} f^{\beta}(2;\, t)\right) \cdot m_{\alpha}^{-1} \partial_1 f^{\alpha}(1;\, t)$$

$$\equiv -\frac{e_{\alpha}}{m_{\alpha}} \bar{\boldsymbol{E}}(\boldsymbol{q}_1;\, t) \cdot \partial_1 f^{\alpha}(1;\, t), \tag{5.3}$$

where

$$\bar{\boldsymbol{E}}(\boldsymbol{q}_1;\, t) \equiv -\nabla_1 \bar{\Phi}(\boldsymbol{q}_1;\, t) \tag{5.4}$$

and:

$$\bar{\Phi}(\boldsymbol{q}_1;\, t) = \sum_{\beta} e_{\beta} \int \mathrm{d}\boldsymbol{q}_2\; \mathrm{d}\boldsymbol{v}_2\; \frac{1}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|} f^{\beta}(\boldsymbol{q}_2,\, \boldsymbol{v}_2;\, t). \tag{5.5}$$

These results have a very simple physical interpretation. We recognize in the second equality (5.3) the gradient of the average interaction potential which, in turn, can be expressed in the form of an electric field $\bar{\boldsymbol{E}}$ deriving from a scalar potential $\bar{\Phi}$, defined in (5.5). The latter has exactly the form of the potential produced by a macroscopic continuous charge distribution σ,

$$\sigma(\boldsymbol{q}_2;\, t) = \sum_{\beta} e_{\beta} \int \mathrm{d}\boldsymbol{v}_2\; f^{\beta}(\boldsymbol{q}_2,\, \boldsymbol{v}_2;\, t). \tag{5.6}$$

More precisely, the potential $\bar{\Phi}(\boldsymbol{q}_1;\, t)$ is a solution of the *Poisson equation*

$$\nabla^2 \bar{\Phi}(\boldsymbol{q}_1;\, t) = -4\pi\sigma(\boldsymbol{q}_1;\, t). \tag{5.7}$$

This effect of the interactions can be described as follows. Each particle (1) feels the action of an average electric field produced by all the other particles in the system; these particles act like a "smeared-out" continuous charge distribution $\sigma(\boldsymbol{q}_1;\, t)$: they lost their discrete, point-like character in the process of averaging. This electric field is *self-consistent* in the following sense: $\bar{\boldsymbol{E}}$ acts

on the distribution function f^α and makes it change in time; but as f^α changes, it modifies, in turn, the potential through (5.5).

We now note that (5.3) has been reduced to the same form as the electric field term in the one-particle Liouville equation (2.20)–(2.21). The latter field $\boldsymbol{E}_0$ is, by definition, an external field, produced by sources outside the plasma: the corresponding potential Φ_0 therefore obeys the *homogeneous* Laplace equation, i.e. eq. (5.7) with $\sigma = 0$. We now define the *total electric field* in the plasma, $\boldsymbol{E}(\boldsymbol{q};\ t)$, as

$$\boldsymbol{E}(\boldsymbol{q};\ t) = \boldsymbol{E}_0(\boldsymbol{q};\ t) + \bar{\boldsymbol{E}}(\boldsymbol{q};\ t). \tag{5.8}$$

It derives from the potential $\Phi = \Phi_0 + \bar{\Phi}$, which obeys the Poisson equation (5.7) (with the same source term σ), with appropriate boundary conditions.

We may thus write the kinetic equation (4.15), omitting the collision term $\mathscr{K}^\alpha$, in the explicit form

$$\partial_t f^\alpha(1;\ t) = -\boldsymbol{v}_1 \cdot \nabla_1 f^\alpha(1;\ t) - \frac{e_\alpha}{m_\alpha}\left(\frac{1}{c}(\boldsymbol{v}_1 \wedge \boldsymbol{B}) + \boldsymbol{E}\right) \cdot \partial_1 f^\alpha(1;\ t), \tag{5.9}$$

which is coupled to the *Poisson equation*

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{q};\ t) = 4\pi \sum_\beta e_\beta \int \mathrm{d}\boldsymbol{v}\, f^\beta(\boldsymbol{q},\ \boldsymbol{v};\ t), \tag{5.10}$$

together with

$$\nabla \wedge \boldsymbol{E}(\boldsymbol{q};\ t) = 0. \tag{5.11}$$

Equation (5.9) is the celebrated *Vlasov equation* (Vlasov 1938) which plays a fundamental role in plasma physics. It is also called the "collisionless kinetic equation" (or, very improperly, the "collisionless Boltzmann equation"). One feature is immediately striking: it has the same form as the one-particle Liouville equation (2.20)–(2.21). But one should never forget the essential difference between these equations. In the Liouville equation, the electric field $\boldsymbol{E}_0$ is *external*, i.e., it is a *prescribed function* of $\boldsymbol{q}$ and t. As a result, the *Liouville equation is linear* in f^α. On the contrary, the "linear" form of (5.9) is only apparent, because here the electric field is to be considered as a second unknown function, along with f^α; the Vlasov equation is *coupled* to the Poisson equation. Hence, the *Vlasov equation is non-linear*. This property is a necessary consequence of the interactions, and clearly appears in the forms

(4.15) and (5.3). It will be seen in Part III of this monograph what an important role this non-linearity plays in the description of plasma turbulence.

From eqs. (5.9)–(5.11) one derives some feeling of uneasiness from the fact that the electric and magnetic fields appear to be treated unsymmetrically. Indeed, $\boldsymbol{E}$ is the total field, determined self-consistently by (5.10)–(5.11), whereas $\boldsymbol{B}$ is purely external. Moreover, $\boldsymbol{E}$ is determined by these equations as a potential, electrostatic field. The origin of this asymmetry is clear: we have described the interactions as purely Coulombic. A more complete treatment should include the full electromagnetic interactions; this, in turn, requires a relativistic theory. This is not an easy matter, when full generality is necessary (Balescu and Kotera 1967, Balescu et al. 1967, Balescu and Poulain 1974). A simple treatment, sufficient for the derivation of the Vlasov equation, is given by Klimontovich (1964). We do not go into the details of the calculation, because the result is easily guessed from the previous discussion. The average velocity of the particles builds up an electric current, which acts as a source of a self-consistent magnetic field $\boldsymbol{B}(\boldsymbol{q};\, t)$ that must be added to the external field in order to produce the total field $\boldsymbol{B}(\boldsymbol{Q};\, t)$.

As a result, the Vlasov equation (5.9) is still valid as it stands, but it is coupled to the full set of *Maxwell equations*,

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{q};\, t) = 4\pi \sum_{\beta} e_{\beta} \int \mathrm{d}\boldsymbol{v}\, f^{\alpha}(\boldsymbol{q},\, \boldsymbol{v};\, t),$$

$$\nabla \wedge \boldsymbol{E}(\boldsymbol{q};\, t) = -c^{-1}\, \partial_t \boldsymbol{B}(\boldsymbol{q};\, t),$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{q};\, t) = 0,$$

$$\nabla \wedge \boldsymbol{B}(\boldsymbol{q};\, t) = c^{-1}\, \partial_t \boldsymbol{E}(\boldsymbol{q};\, t) + (4\pi/c) \sum_{\beta} e_{\beta} \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f^{\alpha}(\boldsymbol{q},\, \boldsymbol{v};\, t). \qquad (5.12)$$

Clearly, if we let $c \to \infty$, we come back to the previous picture. Equation (5.9) and (5.12) form a complete description of the interactions in all cases where the collisions can be neglected.

2.6. *The Landau kinetic equation*

We now turn to the derivation of the collision term in eq. (4.15). This step is not a mere translation of the symbols of (4.11) as in the case of Vlasov equation, but requires new, non-trivial physics. [For a general treatment, see Balescu (1975).]

We start on a formal level. Our goal is the elimination of the correlation function $g^{\alpha\beta}(1, 2; t)$ from the last term of (4.11). For this purpose we must solve one of the truncated versions of the second BBGKY equation: we choose here the simplest case, i.e., the weak coupling version (4.14). This linear, inhomogeneous equation for $g^{\alpha\beta}(1, 2; t)$ is formally solved in terms of a *propagator* $U_{12}^{\alpha\beta}(t)$, which obeys

$$\partial_t U_{12}^{\alpha\beta}(t) - \left(L_1^{\alpha} + L_2^{\beta}\right) U_{12}^{\alpha\beta}(t) = 0, \qquad U_{12}^{\alpha\beta}(0) = I, \tag{6.1}$$

where I is the identity operator. The solution of the initial value problem for (4.14) is then

$$g^{\alpha\beta}(1, 2; t) = \int_0^t \mathrm{d}\tau \; U_{12}^{\alpha\beta}(\tau) L_{12}'^{\alpha\beta} \; f^{\alpha}(1; t-\tau) f^{\beta}(2; t-\tau)$$

$$+ U_{12}^{\alpha\beta}(t) g^{\alpha\beta}(1, 2; 0), \tag{6.2}$$

where $g^{\alpha\beta}(1, 2; 0)$ is the (given) initial condition for the correlation function. Substituting this result into the last term of (4.11), we obtain

$$\mathscr{K}^{\alpha} = \sum_{\beta} \int \mathrm{d}2 \int_0^t \mathrm{d}\tau \; L_{12}'^{\alpha\beta} U_{12}^{\alpha\beta}(\tau) L_{12}'^{\alpha\beta} f^{\alpha}(1; t-\tau) f^{\beta}(2; t-\tau)$$

$$+ \sum_{\beta} \int \mathrm{d}2 \; L_{12}'^{\alpha\beta} U_{12}^{\alpha\beta}(t) g^{\alpha\beta}(1, 2; 0). \tag{6.3}$$

This equation is formally exact (within the weak coupling approximation) and yields a closed equation for $f^{\alpha}(1; t)$. It is called the *master equation* and was originally derived (in a more general context) by Prigogine and Résibois (1961). This equation has some quite unfamiliar properties. A striking feature is its *non-Markovian* character. It is an integro-differential equation in time; hence, the rate of change $\partial_t f^{\alpha}(1; t)$ at time t depends on the integral of $f^{\alpha}(1; t-\tau)$ over the whole past history. Such a structure cannot be expected to lead to the familiar macroscopic equations of hydrodynamics or electrodynamics, which are all Markovian. Moreover, the second term is very strange: it implies that the *rate of change* at time t of $f^{\alpha}(1; t)$ depends on the initial value of the *correlations*, i.e. on the initial preparation of the system. In order to get out of these difficulties, we must exploit more thoroughly the physics of the problem and disentangle the relevant features from the less relevant ones.

We first note that the two-body propagator $U_{12}^{\alpha\beta}(t)$ relates to the unperturbed motion of independent particles. It is well known (or easily checked) that this quantity factorizes as

$$U_{12}^{\alpha\beta}(t) = U_1^{\alpha}(t)U_2^{\beta}(t), \tag{6.4}$$

where the one-particle propagator satisfies the simpler Liouville equation

$$\partial_t U_1^{\alpha}(t) = L_1 U_1^{\alpha}(t). \tag{6.5}$$

Next, we note that the determination of the propagator is equivalent to the solution of the equations of motion of a single particle. Indeed, the Liouville theorem asserts that the phase space density at time t, at the point $(\boldsymbol{q}, \boldsymbol{v})$, is the same as the density at time 0, at the point where the particle was at time $(-t)$:

$$U_1^{\alpha}(t)f^{\alpha}(\boldsymbol{q}_1, \boldsymbol{v}_1; 0) = f^{\alpha}(\boldsymbol{q}_1, \boldsymbol{v}_1; t) = f^{\alpha}(\boldsymbol{q}_1(-t), \boldsymbol{v}_1(-t); 0), \tag{6.6}$$

where $\boldsymbol{q}(-t)$ obeys the "backward" equation of motion,

$$\dot{\boldsymbol{q}}(-t) = -[\boldsymbol{q}(-t), H] = [H, \boldsymbol{q}(-t)] \tag{6.7}$$

with the initial condition

$$\boldsymbol{q}(0) = \boldsymbol{q} \tag{6.8}$$

and similar equations for $\boldsymbol{v}(-t)$. But we know from the discussion in chapter 1 that even this one-particle problem is, in general, very difficult in the presence of inhomogeneous external fields. The feature that saves us and leads to the derivation of an explicit kinetic equation is the fact that *in a plasma there exist several characteristic time- and length-scales* and that these scales may be *widely separated*, at least in some cases. In order to discuss these, we must formulate a certain number of intuitive, qualitative statements or anticipations, which must be checked a posteriori.

We have already mentioned in section 2.4 that the correlation function introduces a characteristic length, the *range of correlation* $r_{c\alpha}$, defined by the fact that $g^{\alpha\beta} \simeq 0$ whenever the relative distance $|\boldsymbol{q}_1 - \boldsymbol{q}_2|$ exceeds $r_{c\alpha}$. On the basis of elementary plasma physics (the Debye theory) it can be argued that $r_{c\alpha}$ is of the order of the *Debye length* of species α,

$$r_{c\alpha} = \lambda_{D\alpha} = \left(\frac{4\pi e_\alpha^2 n_\alpha}{T_\alpha}\right)^{-1/2}, \tag{6.9}$$

where n_α is the number density and T_α is the temperature (expressed in energy units) of species α. To this characteristic length corresponds a characteristic time, the inverse *plasma frequency* of species α,

$$\tau_{c\alpha} = \omega_{P\alpha}^{-1} = \left(\frac{4\pi e_\alpha^2 n_\alpha}{m_\alpha} \right)^{-1/2} . \tag{6.10}$$

Next, we consider the range of the interactions: this poses a problem for the Coulomb forces, which decrease very slowly with distance and can be considered, for many purposes, to have infinite range. Nevertheless, the elementary Debye theory tells us that the statistical correlations produce a screening of these forces for distances larger than $\lambda_{D\alpha}$. (This problem is also discussed at the end of this chapter in Appendix A1). Hence, the *effective range of the interactions* can be estimated as being equal to $r_{c\alpha}$ as well. It then follows that $\tau_{c\alpha}$ estimates the time spent by a typical particle in the sphere of interaction of another, i.e. the *duration of a collision*.

An independent characteristic time is provided by the collision term as a whole. Anticipating the fact that the collisions drive the system towards equilibrium, the collision term can be, very roughly speaking, represented as

$$\mathscr{K}^\alpha \approx \frac{f^\alpha - f_{eq}^\alpha}{\tau_\alpha} . \tag{6.11}$$

We thus introduce the *relaxation time* τ_α of species α, which may be defined as the time over which the distribution function f^α undergoes a significant change under the effect of the collisions. It will be determined more precisely in chapter 4. Associated with it is the *mean free path* of species α, defined as

$$\lambda_{mfp\alpha} = V_{T\alpha}\tau_\alpha , \tag{6.12}$$

where $V_{T\alpha}$ is the thermal velocity of species α,

$$V_{T\alpha} = \left(\frac{2T_\alpha}{m_\alpha} \right)^{1/2} . \tag{6.13}$$

A third, independent characteristic length is associated with the spatial variation of the macroscopic quantities of the system (density, temperature, ...), of the electric and magnetic fields, and also of the distribution function itself.

Assuming that all these lengths are of the same order, we define L_H, the *gradient length scale*, or *hydrodynamic length*, as

$$L_H = \left(\frac{1}{A} \left| \frac{\partial A}{\partial \boldsymbol{q}} \right| \right)^{-1}, \tag{6.14}$$

where A is any one of the above mentioned quantities. We also define a *hydrodynamic time* as

$$\tau_{H\alpha} = L_H / V_{T\alpha}. \tag{6.15}$$

Last, but not least, there is a time scale defined by the *Larmor frequency*,

$$\tau_{B\alpha} = |\Omega_\alpha|^{-1} = \frac{m_\alpha c}{|e_\alpha| B}, \tag{6.16}$$

and, corresponding to it, a length scale defined by the *Larmor radius*:

$$\rho_{L\alpha} = V_{T\alpha} \tau_{B\alpha}. \tag{6.17}$$

The order relations between these scales determines various regimes of evolution. The most familiar one is characterized as follows. The gradient scale length L_H is a macroscopic quantity, typically measured in meters, centimeters,...; it is produced by a macroscopic preparation of the system. At the other extreme, the range of the correlations or of the interactions, $r_{c\alpha}$, is a purely microscopic quantity. For a neutral gas, it is of the order of the size of a molecule, i.e. a few ångstroms. For a plasma, the Debye length can be much larger, depending on the density and the temperature. For laboratory systems (such as a thermonuclear plasma) it does not exceed 10^{-3} cm. As for the mean free path, it is always much longer than the range of the interactions, provided the plasma is weakly coupled; but its relation to the hydrodynamic length is much more variable. In neutral, not too dilute gases, one has $L_H \gg \lambda_{mfp\alpha}$. But in a plasma at high temperature, the mean free path may easily become as large as several meters.

In conclusion, we may assume the following relationships as representing typical situations in a weakly coupled plasma:

$$r_{c\alpha} \ll \lambda_{mfp\alpha} \lesssim L_H, \tag{6.18}$$

together with

$$\tau_{c\alpha} \ll \tau_\alpha \lesssim \tau_{H\alpha}. \tag{6.19}$$

Alternatively, we may have

$$r_{c\alpha} \ll L_H < \lambda_{mfp\alpha}, \tag{6.20}$$

together with

$$\tau_{c\alpha} \ll \tau_{H\alpha} < \tau_\alpha. \tag{6.21}$$

Considering now the magnetic effects, we assume that the magnetic field is sufficiently small in order to ensure the conditions *

$$|\Omega_\alpha| \tau_{c\alpha} < 1, \qquad \frac{r_{c\alpha}}{\rho_{L\alpha}} < 1. \tag{6.22}$$

It may be useful to note that this condition can be satisfied in practice even with very strong magnetic fields, such that

$$\frac{L_H}{\rho_{L\alpha}} \gg 1, \qquad \frac{\lambda_{mfp\alpha}}{\rho_{L\alpha}} \gg 1. \tag{6.23}$$

It should be realized that the first of these conditions precisely defines the applicability of the *drift approximation*, (1.5.26). Forgetting now the details of the rough estimation of the various quantities, we only retain the basic assumptions (6.18)–(6.22) and make the best use of them in deriving an expression for the collision operator.

In kinetic theory, we are mainly interested in the approach to equilibrium. This process takes place on the characteristic time scale τ_α, which is much longer than the duration of the collisions, $\tau_{c\alpha}$. We therefore decide to study the evolution by "smoothing out" the details of the motion that arise on the short time scale $\tau_{c\alpha}$. The resulting "smooth", or "asymptotic" evolution law will be an appropriate description of the phenomena, whenever we are not interested in short-living transient processes. (Note that a similar philosophy underlies the description of the particle motion in the drift approximation; see, in particular, fig. 1.5.4.)

The details of the calculations involved in the implementation of these ideas are given in Appendix A1. The result of this treatment is certainly familiar to

* In reality, condition (6.22) may be somewhat marginal in some large modern tokamaks. Kinetic equations describing "magnetized collisions" have been derived by several authors (see Appendix A1); but this problem will not be studied in this book.

many readers: it is the explicit form of the kinetic equation (4.15), in the weak coupling approximation,

$$\mathscr{K}^{\alpha} = \sum_{\beta} 2\pi e_{\alpha}^{2} e_{\beta}^{2} \ln \Lambda \int \mathrm{d}\boldsymbol{v}_2 \; m_{\alpha}^{-1} \frac{\partial}{\partial v_{1r}} G_{rs}(\boldsymbol{g})$$

$$\times \left(m_{\alpha}^{-1} \frac{\partial}{\partial v_{1s}} - m_{\beta}^{-1} \frac{\partial}{\partial v_{2s}} \right) f^{\alpha}(\boldsymbol{q}_1, \boldsymbol{v}_1; t) f^{\beta}(\boldsymbol{q}_1, \boldsymbol{v}_2; t). \tag{6.24}$$

Here, we use the notation $\boldsymbol{g}$ for the relative velocity vector,

$$\boldsymbol{g} = \boldsymbol{v}_1 - \boldsymbol{v}_2. \tag{6.25}$$

The important *Landau tensor* $G_{rs}(\boldsymbol{a})$ is defined, for an arbitrary vector $\boldsymbol{a}$, as

$$G_{rs}(\boldsymbol{a}) = \frac{a^2 \delta_{rs} - a_r a_s}{a^3}. \tag{6.26}$$

Finally, the *Coulomb logarithm*, $\ln \Lambda$, is defined as

$$\ln \Lambda = \ln \frac{\frac{3}{2}(T_{\mathrm{e}} + T_{\mathrm{i}}) \lambda_{\mathrm{D}}}{Ze^2}. \tag{6.27}$$

Equation (6.24) represents the celebrated *Landau collision term*, that provides the basis of most of the existing works on plasma transport theory. It was first obtained by Landau (1936) from the Boltzmann equation of the kinetic theory of gases, combined with the weak coupling assumption. It has been later derived more rigourously by many authors (Bogoliubov 1946, Balescu 1963, 1975, Klimontovich 1964, 1982) and its properties are discussed in many textbooks on plasma physics (e.g. Montgomery and Tidman 1964, Ichimaru 1974, Krall and Trivelpiece 1986, Golant et al. 1980, Rosenbluth and Sagdeev 1983).

The derivation given in Appendix A1 is intended to pinpoint the steps at which specific assumptions come in. This will help the reader in modifying the kinetic equation whenever some of these assumptions break down.

A fundamental result of this section is the fact that the collision operator $\mathscr{K}^{\alpha}$ appears as an operator acting on $f^{\alpha}(\boldsymbol{q}, \boldsymbol{v}; t)$, i.e. a functional of the distribution function at time t. We have thus proved the existence of a kinetic regime, as announced in (4.15). It appears as an asymptotic regime, valid for times much longer than the duration of a collision. A fuller discussion of the more fundamental aspects of this question can be found in the book by Balescu (1975). For our more restricted purpose, the main point is the existence of a closed kinetic equation for f^{α}: eqs. (4.15) and (6.24).

2.7. *Conservation properties of the collision term*

In the present section we derive some fundamental properties of the Landau collision term (and, for that matter, of *any* acceptable collision term). For this discussion, we need to introduce two lemmas which, although elementary, provide us with some of the most frequently used tools in transport theory.

Till now, we have not yet made any specific assumption about the class of functions to which we must restrict the distribution functions $f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)$. We do not wish to go into sophisticated mathematical discussions at this point; but some simple conditions must be met in order to give a meaning to the forthcoming operations. These conditions originate from the fact that the distribution function is an intermediate quantity in the definition (1.16) of the observable quantities as averages of microscopic dynamical functions. For this definition to make sense, it must be required that the integrals converge. The behaviour of $f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)$ as a function of the position depends on the type of boundary conditions in the physical space (e.g. the presence or absence of a material wall): no general statements can be made. This point will be discussed again in section 3.1. As for the velocity variable $\boldsymbol{v}$, it is clear that each component v_r may vary between $-\infty$ and $+\infty$ (in a non-relativistic theory!). Therefore, an acceptable distribution function must decrease at infinity (as a function of v_x, v_y, v_z) sufficiently fast, in order to ensure the existence of all averages of physical interest. In practice, a decrease of exponential type at infinity meets all the requirements. From this condition, our two lemmas follow immediately:

Lemma A. If $A(\boldsymbol{v})$ is any function of $\boldsymbol{v}$, growing not faster than a polynomial at infinity, we have

$$\int \mathrm{d}\boldsymbol{v} \frac{\partial}{\partial v_r}[A(\boldsymbol{v}) f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)] = 0. \tag{7.1}$$

Lemma B. If $A'(\boldsymbol{v})$ is a function of the same type as $A(\boldsymbol{v})$, we have the important formula of *integration by parts*:

$$\int \mathrm{d}\boldsymbol{v} A'(\boldsymbol{v}) \frac{\partial}{\partial v_r}[A(\boldsymbol{v}) f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)] = -\int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) A(\boldsymbol{v}) \frac{\partial}{\partial v_r} A'(\boldsymbol{v}). \tag{7.2}$$

(Actually, lemma A is but a particular case of lemma B).

We now consider the Landau collision term $\mathscr{K}^\alpha$, defined by (6.24): it is a non-linear operator acting on the distribution functions. More precisely, the non-linearity is quadratic: this means that *binary collisions* are by far dominant over three- or more-particle collisions. In this case, the collision term can be naturally decomposed as

$$\mathscr{K}^\alpha = \sum_{\beta=\mathrm{e,i}} \mathscr{K}^{\alpha\beta}, \tag{7.3}$$

where $\mathscr{K}^{\alpha\beta}$ represents the contribution to the kinetic equation of species α of the collisions of a particle of species α with a particle of species β. Clearly, the term $\mathscr{K}^{\alpha\beta}$ is a functional of the product $f^\alpha f^\beta$.

The collision operator possesses a few properties that are easily understandable. In any collision process, *the number of particles of each species, the total momentum and the total energy must be locally conserved.* In particular, in a binary elastic collision, any amount of momentum and of energy lost by one of the partners at a given point must be gained by the other partner at the same point. Hence, the average number of electrons, the average number of ions, the average total momentum and the average total energy cannot change in time as a result of the collisions. These conservation laws are expressed by the following equations:

$$\int \mathrm{d}\boldsymbol{v}\, \mathscr{K}^\alpha = 0, \quad \alpha = \mathrm{e, i}, \tag{7.4}$$

$$\sum_\alpha m_\alpha \int \mathrm{d}\boldsymbol{v}\, v_r \mathscr{K}^\alpha = 0, \quad r = 1, 2, 3, \tag{7.5}$$

$$\sum_\alpha \tfrac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}\, v^2 \mathscr{K}^\alpha = 0. \tag{7.6}$$

These relations must hold *independently of the form of the distribution functions* f^α involved in the collision term.

The proof of the number conservation property (7.4) is a trivial consequence of lemma A (7.1).

In order to prove the second property (7.5), we write the collision term in a more symmetric form,

$$\mathscr{K}^{\alpha\beta} = A_{\alpha\beta}\int \mathrm{d}\boldsymbol{v}_2 \left(m_\alpha^{-1}\,\partial_{1m} - m_\beta^{-1}\,\partial_{2m}\right) G_{mn}(\boldsymbol{g}) \left(m_\alpha^{-1}\,\partial_{1n} - m_\beta^{-1}\,\partial_{2n}\right)$$

$$\times f^\alpha(1) f^\beta(2), \tag{7.7}$$

with

$$A_{\alpha\beta} \equiv 2\pi e_\alpha^2 e_\beta^2 \ln \Lambda, \qquad f^\alpha(1) \equiv f^\alpha(\boldsymbol{q}_1, \boldsymbol{v}_1; t).$$

Clearly, the added term, starting with $m_\beta^{-1}\,\partial_{2m}$ in the integrand is identically zero, because of lemma A (7.1). Next, we note that the integrand is symmetric under the permutation of the subscripts 1 and 2, together with α and β. Therefore, we may evaluate the expression in (7.5) as

$$\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, m_\alpha v_{1r} \mathscr{K}^\alpha$$

$$= \sum_\alpha \sum_\beta A_{\alpha\beta} \int \mathrm{d}\boldsymbol{v}_1 \int \mathrm{d}\boldsymbol{v}_2\; m_\alpha v_{1r} \left(m_\alpha^{-1}\,\partial_{1m} - m_\beta^{-1}\,\partial_{2m}\right) G_{mn}(\boldsymbol{g})$$

$$\times \left(m_\alpha^{-1}\,\partial_{1n} - m_\beta^{-1}\,\partial_{2n}\right) f^\alpha(1) f^\beta(2)$$

$$= \tfrac{1}{2} \sum_\alpha \sum_\beta A_{\alpha\beta} \int \mathrm{d}\boldsymbol{v}_1 \int \mathrm{d}\boldsymbol{v}_2 (m_\alpha v_{1r} + m_\beta v_{2r}) \left(m_\alpha^{-1}\,\partial_{1m} - m_\beta^{-1}\,\partial_{2m}\right) G_{mn}(\boldsymbol{g})$$

$$\times \left(m_\alpha^{-1}\,\partial_{1n} - m_\beta^{-1}\,\partial_{2n}\right) f^\alpha(1) f^\beta(2)$$

$$= -\tfrac{1}{2} \sum_\alpha \sum_\beta A_{\alpha\beta} \int \mathrm{d}\boldsymbol{v}_1 \int \mathrm{d}\boldsymbol{v}_2 (\delta_{mr} - \delta_{mr}) G_{mn}(\boldsymbol{g})$$

$$\times \left(m_\alpha^{-1}\,\partial_{1n} - m_\beta^{-1}\,\partial_{2n}\right) f^\alpha(1) f^\beta(2) = 0$$

The last step is obtained by an integration by parts.

The third theorem, (7.6), is proved by using the same symmetrization procedure:

$$\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, \tfrac{1}{2} m_\alpha v_1^2 \mathscr{K}^\alpha$$

$$= \sum_\alpha \sum_\beta A_{\alpha\beta} \int \mathrm{d}\boldsymbol{v}_1 \int \mathrm{d}\boldsymbol{v}_2\, \tfrac{1}{2} \left(m_\alpha v_1^2 + m_\beta v_2^2\right) \left(m_\alpha^{-1}\, \partial_{1m} - m_\beta^{-1}\, \partial_{2m}\right) G_{mn}(\boldsymbol{g})$$

$$\times \left(m_\alpha^{-1}\, \partial_{1n} - m_\beta^{-1}\, \partial_{2n}\right) f^\alpha(1) f^\beta(2)$$

$$= -\tfrac{1}{2} \sum_\alpha \sum_\beta A_{\alpha\beta} \int \mathrm{d}\boldsymbol{v}_1 \int \mathrm{d}\boldsymbol{v}_2 (v_{1m} - v_{2m}) G_{mn}(\boldsymbol{g})$$

$$\times \left(m_\alpha^{-1}\, \partial_{1n} - m_\beta^{-1}\, \partial_{2n}\right) f^\alpha(1) f^\beta(2) = 0.$$

The conclusion follows from the very simple and useful property of the Landau tensor $G_{mn}(\boldsymbol{a})$ (eq. 6.26), together with (6.25),

$$G_{mn}(\boldsymbol{a}) a_n = a_m G_{mn}(\boldsymbol{a}) = 0. \tag{7.8}$$

We have thus proved that the Landau collision operator possesses the necessary conservation properties required from a collision operator. It is clear that the form of the distribution functions was not used in proving the conservation laws: these are valid, irrespective of the particular state of the plasma.

It is important to note that some more detailed statements can be proved. Representation (7.3) shows that the collision term $\mathscr{K}^\alpha$ for particles of species α is a sum of two terms, representing, respectively, *like-particle collisions* ($\beta = \alpha$), and *unlike-particle collisions* ($\beta \neq \alpha$). A simple adaptation of the previous calculations leads to the result that *like-particle collisions separately conserve the number, momentum and energy of the particles of species* α. Thus

$$\int \mathrm{d}\boldsymbol{v}_1\, \mathscr{K}^{\alpha\alpha} = 0, \qquad \alpha = \mathrm{e}, \mathrm{i}, \tag{7.9}$$

$$m_\alpha \int \mathrm{d}\boldsymbol{v}_1\, v_{1r} \mathscr{K}^{\alpha\alpha} = 0, \qquad \alpha = \mathrm{e}, \mathrm{i}, \quad r = 1, 2, 3, \tag{7.10}$$

$$\tfrac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}_1\, v_1^2 \mathscr{K}^{\alpha\alpha} = 0, \qquad \alpha = \mathrm{e}, \mathrm{i}. \tag{7.11}$$

Combining these individual conservation laws with the global ones of eqs. (7.4)–(7.6), we obtain the following properties of the *unlike-particle collision operators*:

$$\int \mathrm{d}\boldsymbol{v}_1 \, \mathscr{X}^{\mathrm{ei}} = \int \mathrm{d}\boldsymbol{v}_1 \, \mathscr{X}^{\mathrm{ie}} = 0, \tag{7.12}$$

$$\int \mathrm{d}\boldsymbol{v}_1 \, m_{\mathrm{e}} v_{1r} \mathscr{X}^{\mathrm{ei}} = -\int \mathrm{d}\boldsymbol{v}_1 \, m_{\mathrm{i}} v_{1r} \mathscr{X}^{\mathrm{ie}}, \tag{7.13}$$

$$\int \mathrm{d}\boldsymbol{v}_1 \, \tfrac{1}{2} m_{\mathrm{e}} v_1^2 \mathscr{X}^{\mathrm{ei}} = -\int \mathrm{d}\boldsymbol{v}_1 \, \tfrac{1}{2} m_{\mathrm{i}} v_1^2 \mathscr{X}^{\mathrm{ie}}. \tag{7.14}$$

The importance of these properties for the transport theory will appear clearly in chapter 3.

Another general property of the collision operator is the celebrated *H-theorem*. In view of its particular role in transport theory, a special chapter is devoted to its study (chapter 6; see also chapter 17).

2.8 *The "Lorentz process"*

The Landau collision term has a very nice, symmetric form (7.7). The great symmetry between electrons and ions is, however, misleading. Indeed, the large disparity in mass between the two components, expressed by (1.2), leads to a very different behaviour of the electrons and ions in the collision processes. It is useful to exploit this fact as early as possible in the game: this will lead to substantial simplifications in the forthcoming calculations. The first author who introduced this idea was Braginskii (1965); many others followed him later. Silin (1971) gave a slightly different (but equivalent) version of the method, which we essentially follow here *.

The basic idea consists of noting that, because of their very large mass, the ions have, on the average, velocities much smaller than the electrons. In a rough approximation, one would conceive of the plasma as a collection of electrons moving among, and colliding with a set of infinitely heavy, stationary ions. This picture corresponds to the so-called *Lorentz gas*, a model often studied in the kinetic theory of gases (Lorentz 1905, Sommerfeld 1952, Chapman and Cowling 1952, Balescu 1963, Delcroix 1966). This extreme

* Some of the average quantities, such as the average velocities $\boldsymbol{u}^a$, the temperatures T_a and the pressure tensors $\boldsymbol{\pi}^a$, appearing in this section, will only be defined systematically in chapter 3. Nevertheless, we prefer to discuss these matters in the present chapter, to which they logically belong. Some (well-known) definitions will thus be anticipated.

picture is often too crude for the purpose of transport theory *. It can be taken, however, as a starting point for a kind of "perturbation expansion": we propose to call this expansion the *Lorentz process*, as it takes the Lorentz gas as a reference point.

The Lorentz process concerns only the unlike-particle collisions: in the collision terms $\mathscr{K}^{\alpha\alpha}$, the two collision partners are identical, and the Lorentz model is inapplicable.

We begin our analysis with the *electron–ion collision term*, which we rewrite as

$$\mathscr{K}^{ei} = \frac{2\pi Z^2 e^4 \ln \Lambda}{m_e^2} \partial_{1r} \Phi_r^{ei}, \tag{8.1}$$

with

$$\Phi_r^{ei} = \int d\boldsymbol{v}_2 \, G_{rs}(\boldsymbol{v}_1 - \boldsymbol{v}_2)(\partial_{1s} - \mu \, \partial_{2s}) f^e(\boldsymbol{v}_1) f^i(\boldsymbol{v}_2). \tag{8.2}$$

The parameter μ is the mass ratio defined in (1.2). In these formulae we omit writing the arguments $(\boldsymbol{q}, t)$ in the distribution functions, because they are irrelevant here. Consider now the tensor

$$G_{rs}(\boldsymbol{v}_1 - \boldsymbol{v}_2) \equiv G_{rs}\left[(\boldsymbol{v}_1 - \boldsymbol{u}^e) - (\boldsymbol{v}_2 - \boldsymbol{u}^e)\right], \tag{8.3}$$

where $\boldsymbol{u}^e$ represents the *average* velocity of the electrons. Note that $\boldsymbol{v}_1$ is an *electron velocity* and $\boldsymbol{v}_2$ is an *ion velocity*.

The distribution function of the ions and of the electrons are schematically shown in fig. 8.1. The ion velocity distribution function is very sharply peaked around the average ion velocity $\boldsymbol{u}^i$; its width is of order $(3T_i/m_i)^{1/2}$. On the contrary, the electron distribution function, centred around $\boldsymbol{u}^e$, is much more widely spread out. Thus, on the average,

$$\left\langle |\boldsymbol{v}_1 - \boldsymbol{u}^e|^2 \right\rangle \cong 3\frac{T_e}{m_e},$$

$$\left\langle |\boldsymbol{v}_2 - \boldsymbol{u}^e|^2 \right\rangle = \left\langle |\boldsymbol{v}_2 - \boldsymbol{u}^i - \boldsymbol{u}^{ei}|^2 \right\rangle \cong 3\frac{T_i}{m_i} + |\boldsymbol{u}^{ei}|^2,$$

* The Lorentz gas model turns out to play a quite important role in the transport theory of magnetically confined plasmas, for some specific reasons which will appear in chapters 11 and 14.

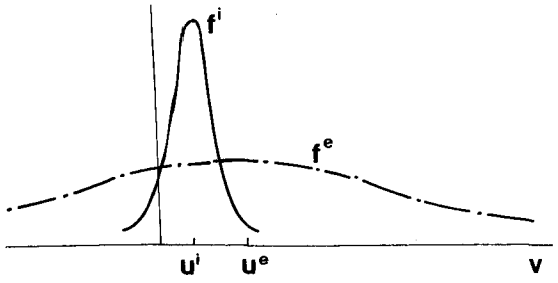

Fig. 8.1. Shape of the ion and electron distribution functions (schematic).

where

$$\boldsymbol{u}^{ei} = \boldsymbol{u}^{e} - \boldsymbol{u}^{i}. \tag{8.4}$$

We then see that, if we assume

$$|\boldsymbol{u}^{ei}| \ll \left(3\frac{T_e}{m_e}\right)^{1/2}, \tag{8.5}$$

and also

$$\frac{T_i}{T_e} \ll \frac{m_i}{m_e} = \mu^{-1}; \tag{8.6}$$

then, over most of the effective range of the variables (determined by the width of the distribution functions), we may consider that

$$|\boldsymbol{v}_1 - \boldsymbol{u}^{e}| \gg |\boldsymbol{v}_2 - \boldsymbol{u}^{e}|. \tag{8.7}$$

Condition (8.5) says that the relative average velocity (which is related to the electric current) should be small compared to the thermal velocity. The second condition (8.6) can be easily satisfied over a wide range of ion temperatures, both smaller and larger than T_e, because μ^{-1} is a very large number.

Our conclusion (8.7) then suggests that if the Landau tensor G_{rs} in eq. (8.3) is expanded in a Taylor series, the latter can be truncated after a finite number of terms:

$$\begin{aligned} &G_{rs}(\boldsymbol{v}_1 - \boldsymbol{u}^{e} - \boldsymbol{v}_2 + \boldsymbol{u}^{e}) \\ &\quad \cong G_{rs}(\boldsymbol{v}_1 - \boldsymbol{u}^{e}) - (v_{2n} - u_n^{e})\, \partial_{1n}\, G_{rs}(\boldsymbol{v}_1 - \boldsymbol{u}^{e}) \\ &\qquad + \tfrac{1}{2}(v_{2n} - u_n^{e})(v_{2m} - u_m^{e})\, \partial_{1n}\, \partial_{1m}\, G_{rs}(\boldsymbol{v}_1 - \boldsymbol{u}^{e}) + \cdots. \end{aligned} \tag{8.8}$$

This result is now substituted into (8.2),

$$\Phi_r^{ei} = \int d\boldsymbol{v}_2 \{ f^i(\boldsymbol{v}_2)[\partial_{1s} f^e(\boldsymbol{v}_1)] - \mu f^e(\boldsymbol{v}_1)[\partial_{2s} f^i(\boldsymbol{v}_2)] \}$$

$$\times [G_{rs}(1) - (v_{2n} - u_n^i - u_n^{ei})\partial_{1n} G_{rs}(1) + \cdots], \tag{8.9}$$

where

$$G_{rs}(1) \equiv G_{rs}(\boldsymbol{v}_1 - \boldsymbol{u}^e)$$

$$= \frac{|\boldsymbol{v}_1 - \boldsymbol{u}^e|^2 \delta_{rs} - (v_{1r} - u_r^e)(v_{1s} - u_s^e)}{|\boldsymbol{v}_1 - \boldsymbol{u}^e|^3}. \tag{8.10}$$

In the form (8.9) the integrations over the velocity $\boldsymbol{v}_2$ are easily performed, by using the definition of the moments, collected in table 3.2.1. The result is

$$\Phi_r^{ei} \cong n_i [G_{rs}(1) + u_n^{ei}\, \partial_{1n} G_{rs}(1)$$

$$+ \tfrac{1}{2}\alpha_{nm}\, \partial_{1n}\, \partial_{1m} G_{rs}(1) + \cdots]\, \partial_{1s} f^e(\boldsymbol{v}_1)$$

$$- \mu n_i [\partial_{1s} G_{rs}(1) + \cdots] f^e(\boldsymbol{v}_1) \tag{8.11}$$

where

$$\alpha_{nm} = \frac{T_i}{m_i}\delta_{rs} + \frac{1}{n_i m_i}\pi_{nm}^i + u_n^{ei} u_m^{ei}. \tag{8.12}$$

Equation (8.11), combined with eq. (8.1), is the expression of the *electron–ion collision operator in the "Lorentz approximation"*.

The interesting feature of this equation is its insensitivity to the details of the ion distribution function. The latter enters only through a few moments of low order (n_i, $\boldsymbol{u}^i$, T_i, π_{nm}^i). Successive approximations in the series contain increasing powers of the mass ratio μ. For consistency with our neglect of terms of order μ, we could actually drop the last term of (8.11) and the last two terms of (8.12). We do keep them, however, in order to show (in section **4**.5) that some important collisional quantities are small, of order μ.

We now turn to the *ion–electron collision operator*:

$$\mathscr{K}^{ie} = \frac{2\pi Z^2 e^4 \ln \Lambda}{m_e^2} \mu\, \partial_{1r}\, \Phi_r^{ie}, \tag{8.13}$$

with

$$\Phi_r^{ie} = \int d\boldsymbol{v}_2\, G_{rs}(\boldsymbol{v}_1 - \boldsymbol{v}_2)\big[\mu f^e(\boldsymbol{v}_2)\, \partial_{1s} f^i(\boldsymbol{v}_1) - f^i(\boldsymbol{v}_1)\, \partial_{2s} f^e(\boldsymbol{v}_2)\big]. \tag{8.14}$$

We may again apply the Lorentz process, but now it is the (unintegrated) velocity $\boldsymbol{v}_1$ which is small. Hence, we write instead of (8.8),

$$\begin{aligned} &G_{rs}(\boldsymbol{v}_1 - \boldsymbol{u}^e - \boldsymbol{v}_2 + \boldsymbol{u}^e) \\ &\quad \cong G_{rs}(2) - (v_{1n} - u_n^e)\, \partial_{2n} G_{rs}(2) \\ &\qquad + \tfrac{1}{2}(v_{1n} - u_n^e)(v_{1m} - u_m^e)\, \partial_{2n}\, \partial_{2m} G_{rs}(2) + \cdots, \end{aligned} \tag{8.15}$$

where

$$G_{rs}(2) = G_{rs}(-\boldsymbol{v}_2 + \boldsymbol{u}^e) = G_{rs}(\boldsymbol{v}_2 - \boldsymbol{u}^e). \tag{8.16}$$

Substituting this expansion into (8.14) we find

$$\begin{aligned} \Phi_r^{ie} = &-L_r f^i(\boldsymbol{v}_1) + M_{rn}\big(v_{1n} - u_n^i - u_n^{ei}\big) f^i(\boldsymbol{v}_1) \\ &- \tfrac{1}{2} N_{rnm}\big(v_{1n} - u_n^i - u_n^{ei}\big)\big(v_{1m} - u_m^i - u_m^{ei}\big) f^i(\boldsymbol{v}_1) + \cdots \\ &+ \mu\big[\hat{L}_{rs}\, \partial_{1s} f^i(\boldsymbol{v}_1) + \cdots\big], \end{aligned} \tag{8.17}$$

where the coefficients are functionals of the electron distribution function, defined as

$$L_r = \int d\boldsymbol{v}_2\, G_{rs}(2) \partial_{2s} f^e(\boldsymbol{v}_2), \quad M_{rn} = \int d\boldsymbol{v}_2 \big[\partial_{2n} G_{rs}(2)\big] \partial_{2s} f^e(\boldsymbol{v}_2),$$

$$N_{rnm} = \int d\boldsymbol{v}_2 \big[\partial_{2n} \partial_{2m} G_{rs}(2)\big] \partial_{2s} f^e(\boldsymbol{v}_2), \quad \hat{L}_{rs} = \int d\boldsymbol{v}_2\, G_{rs}(2) f^e(\boldsymbol{v}_2). \tag{8.18}$$

Note that these coefficients are *not* simple moments of the electron distribution function: their evaluation requires a detailed information about this function.

Equations (8.1), (8.11), (8.13) *and* (8.17) *represent the electron–ion and the ion–electron collision operators in the "Lorentz approximation"*. The illusory

symmetry between ions and electrons has now vanished! It is not difficult to check that the important relations (7.4)–(7.6), which ensure the conservation of the total momentum and of the total energy, are preserved in this approximation.

Appendix 2A.1. Derivation of the collision term

Our starting point is the formally exact expression (6.3) of the collision term. We assume that the effect of the external electric field is negligible in our problem. (More precisely, the electric field produces a negligible acceleration of the particles over the duration of a collision, $\tau_{c\alpha}$). From the condition $L_H \gg r_{c\alpha}$, we conclude that the magnetic field is practically constant over a distance $r_{c\alpha}$. We may therefore evaluate the propagator locally, by using the solution (**1**.5.6)–(**1**.5.9) for a constant field $\boldsymbol{B}$ (pointing in the z-direction). Expressing this solution in Cartesian coordinates we find, for any function F^α,

$$\begin{aligned} &U^\alpha(t)F^\alpha(q_x, q_y, q_z, v_x, v_y, v_z) \\ &\quad = F^\alpha\big(q_x(-t), q_y(-t), q_z(-t), v_x(-t), v_y(-t), v_z(-t)\big) \\ &\quad = F^\alpha\big(q_x - \Omega_\alpha^{-1}v_y(\cos\Omega_\alpha t - 1) - \Omega_\alpha^{-1}v_x \sin\Omega_\alpha t, \\ &\qquad q_y + \Omega_\alpha^{-1}v_x(\cos\Omega_\alpha t - 1) - \Omega_\alpha^{-1}v_y \sin\Omega_\alpha t, q_z - v_z t, \\ &\qquad v_x\cos\Omega_\alpha t - v_y \sin\Omega_\alpha t, v_y\cos\Omega_\alpha t + v_x \sin\Omega_\alpha t, v_z\big). \end{aligned} \tag{A1.1}$$

Consider now the last term in eq. (6.3). We make a change of variables by using, instead of the positions $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, the position $\boldsymbol{q}_1$ and the distance from 1: $\boldsymbol{r} = \boldsymbol{q}_1 - \boldsymbol{q}_2$. We then write

$$\begin{aligned} &\sum_\beta \int \mathrm{d}2\; L_{12}'^{\alpha\beta}U_1^\alpha(t)U_2^\beta(t)g^{\alpha\beta}(\boldsymbol{q}_1, \boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2; 0) \\ &\quad = \sum_\beta \int \mathrm{d}2\; L_{12}'^{\alpha\beta}g^{\alpha\beta}\big(\boldsymbol{q}_1(-t), q_{1x}(-t) - q_{2x}(-t), q_{1y}(-t) - q_{2y}(-t), \\ &\qquad q_{1z} - q_{2z} - (v_{1z} - v_{2z})t, \boldsymbol{v}_1(-t), \boldsymbol{v}_2(-t); 0\big). \end{aligned} \tag{A1.2}$$

We note that, because of the peculiar motion in the magnetic field, the relative distance in the x- and y-directions remains bounded for all times, whereas the distance in the z-direction increases linearly in time. This implies that, after a time

$$t \approx \frac{r_{c\alpha}}{|v_{2z} - v_{1z}|} \approx \frac{r_{c\alpha}}{V_{T\alpha}} \approx \tau_{c\alpha},$$

the relative distance of the particles in the parallel direction becomes larger than the range of the correlations, and therefore,

$$U_{12}^{\alpha\beta}(t) g^{\alpha\beta}(1, 2; 0) \cong 0 \quad \text{for } t > \tau_{c\alpha}. \tag{A1.3}$$

Thus, *the influence of the initial correlations on the evolution process only lasts for a time of order* $\tau_{c\alpha}$. Let us call τ_c the longest one of the two times (τ_{ce}, τ_{ci}) and r_c the corresponding correlation range:

$$\tau_c \equiv \text{Max}(\tau_{ce}, \tau_{ci}), \qquad r_c \equiv \text{Max}(r_{ce}, r_{ci}). \tag{A1.4}$$

From here on, we decide that we are only interested in the study of the *evolution over a time scale much longer than* τ_c,

$$t \gg \tau_c. \tag{A1.5}$$

We are then justified in neglecting the last term of (6.3).

The same physical idea helps in evaluating the remaining term in that equation, although the argument is more elaborate. We first note that, up to terms of higher order in $V^{\alpha\beta}$ (which are negligible in the weak coupling approximation), $f^\alpha(1; t - \tau)$ can be expressed in that term as

$$f^\alpha(1; t - \tau) \cong U_1^\alpha(-\tau) f^\alpha(1; t).$$

Hence, going over to the relative spatial coordinates as before, and using (6.4), (6.6) and the group property $U_1^\alpha(\tau) U_1^\alpha(-\tau) = I$, we find

$$\mathscr{K}^\alpha = \sum_\beta \int dv_2 \int dr \int_0^t d\tau \, m_\alpha^{-1} \partial_1 \cdot \left[\nabla V^{\alpha\beta}(r)\right] \left[\nabla V^{\alpha\beta}(r(-\tau))\right]$$

$$\times \left[U_1^\alpha(\tau) m_\alpha^{-1} \, \partial_1 U_1^\alpha(-\tau)\right.$$

$$\left. - U_2^\beta(\tau) m_\beta^{-1} \, \partial_2 U_2^\beta(-\tau)\right] f^\alpha(1; t) f^\beta(2; t). \tag{A1.6}$$

We note that the factor $V^{\alpha\beta}(\boldsymbol{r})$ differs from zero only when $r < r_c$; moreover, for given r, $V^{\alpha\beta}(\boldsymbol{r}(-\tau))$ differs from zero only when $\tau < \tau_c$ [by the same argument as in eq. (A1.3)]. This means that *only the range of values* $0 < r < r_c$ *and* $0 < \tau < \tau_c$ *contributes effectively to the integrals in* $\mathscr{K}^\alpha$. With this in mind, we evaluate the following expression, taking care with the non-commutation of U_1^α and ∂_1,

$$
\begin{aligned}
& U_1^\alpha(\tau)\, \partial_1 U_1^\alpha(-\tau) f^\alpha(1;\, t) \\
& \quad = \Big\{ \boldsymbol{e}_x \Big[\Omega_\alpha^{-1} \sin \Omega_\alpha \tau\, \nabla_x + \Omega_\alpha^{-1} (\cos \Omega_\alpha \tau - 1)\, \nabla_y \\
& \qquad + \cos \Omega_\alpha \tau\, \partial_x - \sin \Omega_\alpha \tau\, \partial_y \Big] \\
& \quad + \boldsymbol{e}_y \Big[\Omega_\alpha^{-1} \sin \Omega_\alpha \tau\, \nabla_y - \Omega_\alpha^{-1} (\cos \Omega_\alpha \tau - 1)\, \nabla_x \\
& \qquad + \cos \Omega_\alpha \tau\, \partial_y + \sin \Omega_\alpha \tau\, \partial_x \Big] \\
& \quad + \boldsymbol{b} (\tau\, \nabla_z + \partial_z) \Big\} f^\alpha(1;\, t) \\
& \quad \cong (\partial_1 + \tau\, \nabla_1) f^\alpha(1;\, t) + \mathrm{O}(\Omega_\alpha \tau_c).
\end{aligned}
\tag{A1.7}
$$

The last line is obtained by noting that, within the relevant domain, $0 < \tau < \tau_c$, and because of our assumption (6.22), the complete expression can be linearized. The remaining two terms are of quite different order of magnitude, as follows from the obvious estimate

$$
\frac{\tau\, |\nabla f^\alpha|}{|\partial f^\alpha|} \approx \frac{\tau_c}{\tau_H} \ll 1. \tag{A1.8}
$$

Hence, under our assumptions, the whole expression (A1.7) reduces to $\partial_1 f^\alpha(1;\, t)$.

The same argument (6.22) justifies the linear approximation

$$
V^{\alpha\beta}(\boldsymbol{q}_1(-\tau) - \boldsymbol{q}_2(-\tau)) \approx V^{\alpha\beta}(\boldsymbol{r} - \boldsymbol{g}\tau) + \mathrm{O}(\Omega_\alpha \tau_c), \tag{A1.9}
$$

where we introduced the relative velocity

$$
\boldsymbol{g} = \boldsymbol{v}_1 - \boldsymbol{v}_2. \tag{A1.10}
$$

A final important simplification occurs in considering the *localization* of the two distribution functions of the integrand in (A1.6):

$$
\begin{aligned}
& f^{\alpha}(1;\, t) f^{\beta}(2;\, t) \\
& \quad = f^{\alpha}(\boldsymbol{q}_1,\, \boldsymbol{v}_1;\, t) f^{\beta}(\boldsymbol{q}_1 + \boldsymbol{r},\, \boldsymbol{v}_2;\, t) \\
& \quad \approx f^{\alpha}(\boldsymbol{q}_1,\, \boldsymbol{v}_1;\, t)\left[f^{\beta}(\boldsymbol{q}_1,\, \boldsymbol{v}_2;\, t) + \boldsymbol{r} \cdot \nabla_1 f^{\beta}(\boldsymbol{q}_1,\, \boldsymbol{v}_2;\, t) + \cdots \right] \\
& \quad \approx f^{\alpha}(\boldsymbol{q}_1,\, \boldsymbol{v}_1;\, t) f^{\beta}(\boldsymbol{q}_1,\, \boldsymbol{v}_2;\, t)\left[1 + \mathrm{O}(r_{\mathrm{c}}/L_{\mathrm{H}})\right]. \qquad \text{(A1.11)}
\end{aligned}
$$

Collecting now all these simplifications, we reduce the collision term (A1.6) to the form

$$
\begin{aligned}
\mathscr{K}^{\alpha} \cong \sum_{\beta} \int \mathrm{d}\boldsymbol{v}_2 \int \mathrm{d}\boldsymbol{r} \int_0^{\infty} \mathrm{d}\tau\; m_{\alpha}^{-1}\, \partial_1 \cdot \left\{ \left[\nabla V^{\alpha\beta}(\boldsymbol{r}) \right] \left[\nabla V^{\alpha\beta}(\boldsymbol{r} - \boldsymbol{g}\tau) \right] \right\} \\
\times \left(m_{\alpha}^{-1}\, \partial_1 - m_{\beta}^{-1}\, \partial_2 \right) f^{\alpha}(\boldsymbol{q}_1,\, \boldsymbol{v}_1;\, t) f^{\beta}(\boldsymbol{q}_1,\, \boldsymbol{v}_2;\, t), \qquad \text{(A1.12)}
\end{aligned}
$$

or, more briefly,

$$
\begin{aligned}
\mathscr{K}^{\alpha} = \sum_{\beta} \int \mathrm{d}\boldsymbol{v}_2 \int \mathrm{d}\boldsymbol{r} \int_0^{\infty} \mathrm{d}\tau\; m_{\alpha}^{-1}\, \partial_{1r} T_{rs}^{\alpha\beta}(\boldsymbol{g}) \\
\times \left(m_{\alpha}^{-1}\, \partial_{1s} - m_{\beta}^{-1}\, \partial_{2s} \right) f^{\alpha}(\boldsymbol{q}_1,\, \boldsymbol{v}_1;\, t) f^{\beta}(\boldsymbol{q}_1,\, \boldsymbol{v}_2;\, t), \qquad \text{(A1.13)}
\end{aligned}
$$

in which the tensor $T_{rs}^{\alpha\beta}(\boldsymbol{g})$ is defined by comparison with (A1.12). The domain of integration over τ has been extended to infinity: this is admissible, because as soon as $\tau > \tau_{\mathrm{c}}$, the integrand in the added interval $t < \tau < \infty$ is practically zero.

We note that, up to terms of order $(r_{\mathrm{c}}/L_{\mathrm{H}})$ and $(r_{\mathrm{c}}/\rho_{\mathrm{L}})$, the effects of the spatial inhomogeneities and of the external field have been eliminated from the collision term. One important consequence is that the dependence on $\boldsymbol{r}$ and on τ is now confined to the potential energy factors (and has disappeared from the distribution functions). This is a considerable simplification in the structure of the kinetic equation: the integrations over $\boldsymbol{r}$ and τ can be performed independently of the distribution functions. For this purpose, it is

useful to introduce the Fourier representation of the interaction potential,

$$V^{\alpha\beta}(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{k}\, \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\tilde{V}^{\alpha\beta}(\boldsymbol{k}). \tag{A1.14}$$

In the special case of the Coulomb potential (4.2),

$$\tilde{V}^{\alpha\beta}(\boldsymbol{k}) = \frac{1}{2\pi^2}\frac{e_\alpha e_\beta}{k^2}. \tag{A1.15}$$

It is easily checked that

$$\tilde{V}^{\alpha\beta}(\boldsymbol{k}) = \tilde{V}^{\alpha\beta}(-\boldsymbol{k}) = \tilde{V}^{\alpha\beta}(k). \tag{A1.16}$$

The integrations in the tensor $T_{rs}^{\alpha\beta}(\boldsymbol{g})$ are now easily performed. (We provisionally drop all the superscripts α, β):

$$\begin{aligned}
\boldsymbol{T}(\boldsymbol{g}) &= \int \mathrm{d}\boldsymbol{r} \int_0^\infty \mathrm{d}\tau [\nabla V(\boldsymbol{r})][\nabla V(\boldsymbol{r}-\boldsymbol{g}\tau)] \\
&= \int \mathrm{d}\boldsymbol{r} \int_0^\infty \mathrm{d}\tau \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}' \\
&\quad \times \exp[\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r} - \mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{g}\tau](\mathrm{i}\boldsymbol{k})(\mathrm{i}\boldsymbol{k}')\tilde{V}(\boldsymbol{k})\tilde{V}(\boldsymbol{k}') \\
&= (2\pi)^3 \int_0^\infty d\tau \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{k}\boldsymbol{k}\tilde{V}^2(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{g}\tau) \\
&= 8\pi^4 \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{k}\boldsymbol{k}\tilde{V}^2(k)\delta(\boldsymbol{k}\cdot\boldsymbol{g}).
\end{aligned} \tag{A1.17}$$

This can be further integrated, by first choosing a reference frame in which the vector $\boldsymbol{g}$ points along the z-axis, and using spherical coordinates in this frame

$$\begin{aligned}
\boldsymbol{T}(\boldsymbol{g}) &= 8\pi^4 \int_0^\infty \mathrm{d}k\, k^4 \tilde{V}^2(k) \int_{-1}^1 \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\varphi \\
&\quad \times \delta(kg\cos\theta) \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \\
&= (\boldsymbol{I} - \boldsymbol{e}_z\boldsymbol{e}_z)T.
\end{aligned} \tag{A1.18}$$

Indeed, all non-diagonal terms vanish, and so does the diagonal z–z element, because of the δ-function. The scalar $T^{\alpha\beta}$ is

$$T^{\alpha\beta} \equiv T_{xx}^{\alpha\beta} = 8\pi^5 \int_0^\infty \mathrm{d}k\, k^4 \left[\tilde{V}^{\alpha\beta}(k)\right]^2 \int_{-1}^{1} \mathrm{d}\mu (1-\mu^2)\delta(kg\mu)$$

$$= \frac{A_{\alpha\beta}}{g}, \tag{A1.19}$$

with

$$A_{\alpha\beta} = \int_0^\infty \mathrm{d}k\, k^3 \left[\tilde{V}^{\alpha\beta}(k)\right]^2. \tag{A1.20}$$

The tensor $\boldsymbol{T}^{\alpha\beta}$ can now be evaluated in an arbitrary reference frame, in which its components are

$$T_{rs}^{\alpha\beta}(\boldsymbol{g}) = \frac{A_{\alpha\beta}}{g}\left(\delta_{rs} - \frac{g_r g_s}{g^2}\right). \tag{A1.21}$$

We reach the very remarkable conclusion that, for a weakly coupled system, *the nature of the interaction potential enters the collision operator only through the constant* $A_{\alpha\beta}$.

If we now substitute the Coulomb potential (A1.15) into (A1.20), we meet with a well-known difficulty:

$$A_{\alpha\beta}^{\mathrm{Coul}} = 2\pi e_\alpha^2 e_\beta^2 \int \mathrm{d}k \left(k^3/k^4\right) = 2\pi e_\alpha^2 e_\beta^2 \{\ln k\}_0^\infty. \tag{A1.22}$$

The integral diverges logarithmically at both limits. There is a rather obvious explanation for this difficulty.

(a) The divergence for large k corresponds to a divergence for small distances. Clearly, *for very close encounters, the assumption of weak coupling breaks down.*

(b) The small-k divergence corresponds to large distances. It is well-known that the Coulomb potential falls off very slowly; as a result, at large distances, a given particle 1 feels the effect of its "binary-collision partner" 2, but also of many others, located at comparable distances. Thus, because of the long range of the Coulomb potential, *it is the binary-collision approximation, incorporated in* eq. (4.14), *that breaks down.*

Problem b has been solved rigourously. The idea is to introduce collective effects into the description: This is realized by using the more complete

equation (4.13), rather than (4.14), for the elimination of the correlation function. We cannot go here into the details of the calculation (Balescu 1960, 1963, 1975, Lenard 1960), but we can give the final result, which is remarkably simple. It consists of replacing the tensor $\boldsymbol{T}^{\alpha\beta}$ in (A1.17) by

$$\boldsymbol{T}_{\mathrm{BL}}^{\alpha\beta} = 8\pi^4 \int \mathrm{d}\boldsymbol{k}\, \boldsymbol{kk}\, \delta(\boldsymbol{k}\cdot\boldsymbol{g}) \frac{\left[\tilde{V}^{\alpha\beta}(k)\right]^2}{|\epsilon(\boldsymbol{k},\, \boldsymbol{k}\cdot\boldsymbol{v}_1)|^2} \tag{A1.23}$$

where $\epsilon(\boldsymbol{k},\, \boldsymbol{k}\cdot\boldsymbol{v})$ is the *complex dielectric function*, well known from the theory of the linearized Vlasov equation

$$\epsilon(\boldsymbol{k},\, \boldsymbol{k}\cdot\boldsymbol{v}) = 1 + \sum_{\beta} \frac{4\pi e_\beta^2}{m_\beta k^2} \int \mathrm{d}\boldsymbol{v}_2 \, \frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{v}_2 - \mathrm{i}\eta} \, \boldsymbol{k}\cdot\partial_2 f^\beta(2;\, t), \tag{A1.24}$$

where η is an infinitesimal positive number.

This equation shows that the main effect of the collective interactions is to build up a "*dynamical polarization*" that screens off the long-range Coulomb potential. This screening is a self-consistent effect, as seen from (A1.24), where the dielectric function is evaluated with the instantaneous distribution function. As a result the *Balescu–Lenard collision term*, which is obtained by combining (A1.23) with (A1.13), is a highly non-linear equation in f^α, as should be expected for a many-body process. A rough, but often sufficient approximation results from evaluating the dielectric function with a Maxwellian distribution function and taking its static ($\boldsymbol{v} = 0$) limit. The resulting effective potential $\tilde{V}^{\alpha\beta}/\epsilon$ is then the familiar *Debye potential* $\tilde{V}_{\mathrm{D}}^{\alpha\beta}(k)$,

$$\tilde{V}_{\mathrm{D}}^{\alpha\beta}(k) = \frac{e_\alpha e_\beta}{2\pi^2} \frac{1}{k^2 + \kappa_{\mathrm{D}}^2}, \tag{A1.25}$$

where κ_{D} is the inverse Debye length,

$$\kappa_{\mathrm{D}} = \lambda_{\mathrm{D}}^{-1}, \tag{A1.26}$$

with λ_{D} defined by (4.8). The inverse transform of the potential is:

$$V_{\mathrm{D}}^{\alpha\beta}(r) = e_\alpha e_\beta \frac{\exp(-\kappa_{\mathrm{D}} r)}{r}. \tag{A1.27}$$

With the Debye potential, the convergence of (A1.10) is ensured at large r (or small k). However, we still have the problem of the short-distance

behaviour. In order to tackle this problem, one should use the complete equation (4.12) for the correlations; but this leads to untractable mathematical difficulties. The problem is therefore usually treated in a pragmatic way, by introducing an upper cut-off at a value $k = \kappa_{M\alpha}$ in the integral (A1.20). As the divergence is logarithmic, the result is not very sensitive to the exact choice of $\kappa_{M\alpha}$. On a physical basis, one chooses for $\kappa_{M\alpha}$ the inverse distance of closest approach, at which a collision produces a 90° deflection. This turns out to be

$$\kappa_{M\alpha} = \frac{3T_\alpha}{e_\alpha^2}. \tag{A1.28}$$

In conclusion, we evaluate the constant $A_{\alpha\beta}$ in (A1.20) as

$$A_{\alpha\beta} = 8\pi^5 \int_0^{\kappa_M} \mathrm{d}k\, k^3 \left[\tilde{V}_{\mathrm{D}}^{\alpha\beta}(k)\right]^2 = \pi e_\alpha^2 e_\beta^2 \left(\frac{\kappa_{\mathrm{D}}^2}{\kappa_M^2 + \kappa_{\mathrm{D}}^2} - 1 + \ln \frac{\kappa_M^2 + \kappa_{\mathrm{D}}^2}{\kappa_{\mathrm{D}}^2} \right).$$

This result can be further simplified, on the basis that $\kappa_{\mathrm{M}} \gg \kappa_{\mathrm{D}}$; as a result, one only retains the leading, logarithmic term

$$A_{\alpha\beta} = 2\pi e_\alpha^2 e_\beta^2 \ln \Lambda$$

in agreement with eq. (6.24) in the main text. In order to have a unique value for all couples of particles (e–e, e–i, i–i) we introduced an averaging over the species (Trubnikov 1965) in defining the unique Coulomb logarithm (6.27). Some authors (e.g. Braginskii (1965) consider semi-empirical corrections to $\ln \Lambda$ under various conditions of temperature and density; we do not discuss these minor points here.

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3

The macroscopic description of a plasma

3.1. *Local distribution functions*

In kinetic theory, as shown in chapter 2, the statistical state of the quiescent plasma at the microscopic level is completely defined by the specification of the (one-particle) *distribution functions* $f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)$, $(\alpha = \mathrm{e}, \mathrm{i})$. Their precise interpretation is the following: $f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{v}$ is the number of particles of species α located, at time t, in a volume element of size $\mathrm{d}q_1\, \mathrm{d}q_2\, \mathrm{d}q_3\, \mathrm{d}v_1\, \mathrm{d}v_2\, \mathrm{d}v_3$ centred around the point of coordinates $\boldsymbol{q}$, $\boldsymbol{v}$ in the six-dimensional phase space; $\boldsymbol{q}$ denotes the position coordinate of the particle and $\boldsymbol{v}$ its velocity.

The qualification *quiescent* denotes a plasma which is not turbulent. In the latter case, the distribution functions do not provide a complete description of the plasma. This case will be treated in Part III of this work.

Before proceeding, we define precisely some concepts which are of great importance in transport theory.

A system is said to be *homogeneous* (or *spatially uniform*) when all its local properties are the same at all points in space. In other words, a homogeneous system is *invariant under spatial translations*. It is then easily shown that the one-particle distribution functions $f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)$ must be independent of the coordinate $\boldsymbol{q}$ of the particle. They can then always be written in the form

$$f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) = n_\alpha\, \varphi^\alpha(\boldsymbol{v}; t). \qquad [\mathrm{HOM}] \tag{1.1}$$

Here n_α denotes the (constant) number density of the particles of species α,

$$n_\alpha = \frac{N_\alpha}{V}. \qquad [\mathrm{HOM}] \tag{1.2}$$

Remember that a real plasma is always *globally neutral* (see eq. 2.1.3), i.e. the *total* negative charge of the electrons compensates the total positive charge of the ions,

$$N_\mathrm{e} = ZN_\mathrm{i}. \tag{1.3}$$

For a homogeneous system, eqs. (1.2) and (1.3) imply that the plasma is also *locally neutral*, i.e. the charge compensation holds at each point:

$$n_e = Zn_i \qquad [\text{HOM}] \tag{1.4}$$

The function $\varphi^\alpha(\boldsymbol{v};\ t)$ in (1.1) is the *velocity distribution* of the particles of species α. From (2.2.19) and (1.1) follows the normalization

$$\int \mathrm{d}\boldsymbol{v}\ \varphi^\alpha(\boldsymbol{v};\ t) = 1. \tag{1.5}$$

If the system is not only homogeneous, but also *isotropic* in velocity space, the velocity distribution function only depends on the length of the vector $\boldsymbol{v}$:

$$\varphi^\alpha(\boldsymbol{v};\ t) = \varphi^\alpha(v;\ t). \qquad [\text{ISOTR}] \tag{1.6}$$

Finally, the system is said to be in a *stationary* (or *steady*) state whenever its distribution functions are independent of time.

We have recalled in section 2.1 [see also Balescu (1975)] that to each microscopic dynamical function $b^\alpha(\boldsymbol{q}, \boldsymbol{v};\ \boldsymbol{x})$ corresponds a macroscopic dynamical function $B^\alpha(\boldsymbol{x},\ t)$, obtained by the process of statistical averaging,

$$B^\alpha(\boldsymbol{x},\ t) = \int \mathrm{d}\boldsymbol{q}\ \mathrm{d}\boldsymbol{v}\ b^\alpha(\boldsymbol{q}, \boldsymbol{v};\ \boldsymbol{x}) f^\alpha(\boldsymbol{q}, \boldsymbol{v};\ t). \tag{1.7}$$

Here $b^\alpha(\boldsymbol{q}, \boldsymbol{v};\ \boldsymbol{x})$ denotes any function defined on the phase space, i.e. any function of position $\boldsymbol{q}$ and of velocity $\boldsymbol{v}$ of a point-particle of species α. This function may depend, moreover, parametrically on the variable $\boldsymbol{x}$ which denotes the coordinates of a (geometrical) point in the three-dimensional physical space. In other words, $\boldsymbol{x}$ is *not* to be considered as a coordinate of the phase space. The macroscopic dynamical functions are, generally speaking, *fields*, i.e. quantities defined *locally* at each point $\boldsymbol{x}$ of the physical space and at each time t.

Equation (1.7) provides the basic link between microscopic and macroscopic physics. This general relation takes a simpler form when applied to a special class of dynamical functions of particular interest in hydrodynamics and electrodynamics. The quantities which enter the latter domains of physics are all of the nature of *local densities* (e.g. mass or charge densities, momentum or electrical current densities, etc.). To such macroscopic densities correspond microscopic phase space functions which have a common characteristic structure:

$$b^\alpha(\boldsymbol{q}, \boldsymbol{v};\ \boldsymbol{x}) = \beta^\alpha(\boldsymbol{v})\ \delta(\boldsymbol{q} - \boldsymbol{x}). \tag{1.8}$$

In order to understand this formula, consider the simple case of the mass density $\rho(\boldsymbol{x}, t)$, defined macroscopically as the ratio of the mass contained in a region centered around $\boldsymbol{x}$, to the volume V of this region, in the limit as $V \to 0$. In order to translate this definition to the microscopic, discontinuous world, we consider again a region of volume V around point $\boldsymbol{x}$. If the position $\boldsymbol{q}$ of a particle is located within this region, the particle (of species α) contributes a quantity (m_α/V) to the density; if it is outside this region, it contributes zero. As we let the volume V go to zero, we see that the particle gives a non-vanishing contribution to the density only if its position coincides exactly with point $\boldsymbol{x}$; in this case, the contribution (m_α/V) is infinite as $V \to 0$. Hence, the microscopic mass density is $m_\alpha\, \delta(\boldsymbol{q} - \boldsymbol{x})$, which is indeed of the form (1.8).

The average of any microscopic local density of the form (1.8) is given, through (1.7), by

$$B^\alpha(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{v}\, \beta^\alpha(\boldsymbol{v})\, f^\alpha(\boldsymbol{v};\, \boldsymbol{x}, t). \tag{1.9}$$

For all quantities of the form (1.8), the non-trivial part of the averaging concerns only the velocity space. The distribution function $f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t)$ has been transformed in this process into a *local velocity distribution function*,

$$f^\alpha(\boldsymbol{v};\, \boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{q}\, \delta(\boldsymbol{q} - \boldsymbol{x})\, f^\alpha(\boldsymbol{q}, \boldsymbol{v};\, t), \tag{1.10}$$

which depends *parametrically* on the physical space and time coordinates: $(\boldsymbol{x}, t)$. It may be said that $f^\alpha(\boldsymbol{v};\, \boldsymbol{x}, t)$ is a *functional field*: to each point in space and to each time it assigns a function of $\boldsymbol{v}$. Although the difference between ordinary and localized distribution functions may seem trivial in this case, it is worth insisting on their different nature *. We will see later that the "localization" is not always expressed as simply as in (1.10).

Remembering the discussion at the beginning of section **2**.7, we now see that, for the calculation of any macroscopic *density*, the averaging over the position $\boldsymbol{q}$ is trivial. As a result, we do not have to worry about the behaviour of $f^\alpha(\boldsymbol{q}, \boldsymbol{v};\, t)$ at the boundary of the plasma. This agrees with the idea that, for a very large system, the local microscopic state is not influenced by the

* It would be appropriate to use different notations for the ordinary distribution functions [e.g. $f^\alpha(\boldsymbol{q}, \boldsymbol{v};\, t)$] and for the local distribution functions [e.g. $\bar{f}^\alpha(\boldsymbol{v};\, \boldsymbol{x}, t)$]. However, we prefer not to burden the notations, especially because bars and tildas will be necessarily used later on with a different meaning. The enumeration of the variables, their position to the left or to the right of the semicolon, or simply the context will clearly indicate the type of the distribution function.

boundary. This is a fundamental concept of statistical mechanics, connected to the *thermodynamic limit*. A detailed discussion is found in the book by Balescu (1975).

Note that the influence of the boundaries is transferred to the *macroscopic* problem, i.e. to the solution of the differential equations governing the macroscopic fields describing the plasma.

To summarize this discussion, we note that the local densities of interest in hydrodynamics and in electrodynamics are expressed by (1.9) as *averages in velocity space*, evaluated with the *local distribution functions* $f^\alpha(\boldsymbol{v};\ \boldsymbol{x},\ t)$.

3.2. *Macroscopic quantities of a plasma*

Whenever the dynamical function $\beta^\alpha(\boldsymbol{v})$ in (1.8) is a *polynomial* in the three components of the velocity, i.e.

$$\beta^\alpha(\boldsymbol{v}) = \sum_{r_1=0}^{s_1} \sum_{r_2=0}^{s_2} \sum_{r_3=0}^{s_3} a^\alpha_{r_1 r_2 r_3}\, v_x^{r_1} v_y^{r_2} v_z^{r_3}, \tag{2.1}$$

with $s_1 + s_2 + s_3 = s$, where $a^\alpha_{r_1 r_2 r_3}$ are real constant coefficients, the corresponding average will be called a *moment (of order s)* of the local distribution function $f^\alpha(\boldsymbol{v};\ \boldsymbol{x},\ t)$. In the remainder of this book, we shall always tacitly assume that the distribution functions considered in the theory possess moments of any finite order (see the discussion in section **2**.7).

We now discuss the most important moments of transport theory. The simplest moments are the local *number densities* of the particles of species α,

$$n_\alpha(\boldsymbol{x},\ t) = \int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{v};\ \boldsymbol{x},\ t). \tag{2.2}$$

In a fully ionized plasma (when ionization and recombination processes are negligible) the number densities $n_\alpha(\boldsymbol{x},\ t)$ are densities of *conserved quantities* (which does not mean that they are independent of time!: see section 3.4). In a homogeneous plasma they reduce to the constants n_α defined in (1.2).

The next simplest moment is the *average local velocity* of the particles of species α: $\boldsymbol{u}^\alpha(\boldsymbol{x},\ t)$, defined via the *flux of particles* of species α, $\boldsymbol{\Gamma}^\alpha(\boldsymbol{x},\ t)$,

$$\boldsymbol{\Gamma}^\alpha(\boldsymbol{x},\ t) \equiv n_\alpha(\boldsymbol{x},\ t)\ \boldsymbol{u}^\alpha(\boldsymbol{x},\ t) = \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f^\alpha(\boldsymbol{v};\ \boldsymbol{x},\ t). \tag{2.3}$$

It is clear that the particle fluxes $\boldsymbol{\Gamma}^\alpha$ are fundamental quantities for characterizing the efficiency of a plasma confinement device.

The third quantity of interest is the *total average kinetic energy density* o the particles of species α, $n_\alpha(x, t)\ \mathscr{E}_\alpha(x, t)$,

$$n_\alpha(x, t)\ \mathscr{E}_\alpha(x, t) = \tfrac{1}{2}m_\alpha \int \mathrm{d}v\, v^2 f^\alpha(v; x, t). \tag{2.4}$$

This quantity, which is the average of a nonlinear function of v, is conveniently split into two terms of different physical meaning. A fluid of particles α, moving with an average velocity $u^\alpha(x, t)$, possesses a *macroscopic kinetic energy density* $\frac{1}{2}m_\alpha n_\alpha(x, t)\,|u^\alpha(x, t)|^2$. This, however, does not account for the total average kinetic energy density. Indeed, the individual particles move with all possible velocities. Although the average deviation of the individual velocities from u^α vanishes by definition,

$$\int \mathrm{d}v[v - u^\alpha(x, t)]\, f^\alpha(v; x, t) = 0, \tag{2.5}$$

the average of the squared deviation does not. This extra contribution of the random motion to the total energy density is identified with the *internal energy density* (or *thermal energy density*) of the particles of species α. We thus write

$$n_\alpha(x, t)\ \mathscr{E}_\alpha(x, t) = \tfrac{1}{2}m_\alpha n_\alpha(x, t)\,|u^\alpha(x, t)|^2 + n_\alpha(x, t)\ \varepsilon_\alpha(x, t), \tag{2.6}$$

and we easily derive the definition

$$n_\alpha(x, t)\ \varepsilon_\alpha(x, t) = \tfrac{1}{2}m_\alpha \int \mathrm{d}v\ |v - u^\alpha(x, t)|^2\ f^\alpha(v; x, t). \tag{2.7}$$

It must be stressed at this point that the identification of the internal energy (of each species) with the quantity $n_\alpha\varepsilon_\alpha$ defined by (2.7) presupposes that the average potential energy due to the interaction forces between the particles is negligible compared to their kinetic energy. In this case, the plasma behaves *thermodynamically* as an *ideal system*. This assumption is justified for *quiescent plasmas* of low density and/or high temperature, i.e. when the condition (2.4.10) of *weak coupling* is satisfied.

We now introduce two additional, related quantities. It is well known that in thermodynamic equilibrium there exists a simple relation between the internal energy density and the (scalar) pressure of an ideal gas: $P_\alpha = \frac{2}{3}n_\alpha\varepsilon_\alpha$. It will be seen in section 3.4 that this relation holds also in a non-equilibrium plasma; hence, we identify the *pressure of the particles of species* α as

$$P_\alpha(x, t) = \tfrac{1}{3}m_\alpha \int \mathrm{d}v\ |v - u^\alpha(x, t)|^2\ f^\alpha(v; x, t). \tag{2.8}$$

Another quantity, which is much less orthodox from the thermodynamic point of view, is the *temperature of the particles of species* α. The strict notion of temperature is associated with thermodynamic equilibrium: it is *not* the average of a microscopic dynamical function (the idea of temperature of a single particle is meaningless), but rather a property characterizing *globally* a particular equilibrium state of the system [see the discussion in sections 2.2, 2.4 in Balescu (1975)]. It so happens that, for an ideal gas in equilibrium, there is a well-known relation between the internal energy per particle ε_α, and the temperature $\bar{T}_\alpha$,

$$\varepsilon_\alpha = \tfrac{3}{2} k_B \bar{T}_\alpha, \qquad \text{[EQUIL]} \tag{2.9}$$

where k_B is the Boltzmann constant. Moreover, in thermodynamic equilibrium, all the species have the same temperature.

This relation will be modified in two ways. The first one is rather trivial and is a simple matter of notation. As the temperature will appear in all the forthcoming formulae in the combination $k_B \bar{T}_\alpha$, we will spare a lot of space by introducing a symbol T_α defined as

$$T_\alpha \equiv k_B \bar{T}_\alpha. \tag{2.10}$$

Thus, T_α is simply the temperature, measured in energy units (ergs, joules or, more frequently, electron-volts), rather than in degrees Kelvin.

A bolder extrapolation consists of extending (2.9) out of equilibrium, by identifying the temperature as $\frac{2}{3} n_\alpha^{-1}$ times the right-hand side of (2.7). The quantity obtained in this way does not have, in general, the properties of the thermodynamic temperature. It is merely, up to a factor, an alternative name for the nonequilibrium internal energy density. We shall follow this universal practice of plasma physics and *define* the *temperature of the particles of species* α by

$$n_\alpha(x, t)\, T_\alpha(x, t) = \tfrac{1}{3} m_\alpha \int \mathrm{d}v\, |v - u^\alpha(x, t)|^2\, f^\alpha(v; x, t)$$

$$= P_\alpha(x, t). \tag{2.11}$$

This concept will be further discussed below.

The moments discussed up to this point are simple and convenient quantities. The set of quantities n_α, u^α, T_α will be called the *plasmadynamical variables*. They constitute the so-called *two-fluid description of the plasma*, widely used in many texts (see, e.g. Krall and Trivelpiece 1986, Golant et al. 1980, Rosenbluth and Sagdeev 1983, Chen 1984). But the quantities which are

most directly observed are those which characterize the *hydrodynamical and electrodynamical state* of the plasma. They are combinations of the two-fluid moments, the former involving the masses of the particles, the latter their charges. This set of quantities constitutes the so-called *one-fluid description of the plasma*.

The *mass density* of the plasma, $\rho(x, t)$ is naturally defined as

$$\rho(x, t) = \sum_{\alpha} m_{\alpha} n_{\alpha}(x, t). \tag{2.12}$$

The *centre-of-mass velocity* (or *barycentric velocity*) $u(x, t)$, is introduced through

$$\rho(x, t)\, u(x, t) = \sum_{\alpha} m_{\alpha} n_{\alpha}(x, t)\, u^{\alpha}(x, t). \tag{2.13}$$

The product $\rho(x, t)\, u(x, t)$ has the meaning of the *total momentum density* of the plasma.

In a similar way, we define the main electrodynamical moments. The electrical *charge density* $\sigma(x, t)$ is given by

$$\sigma(x, t) = \sum_{\alpha} e_{\alpha} n_{\alpha}(x, t). \tag{2.14}$$

The *electric current density* $j(x, t)$ is defined as *

$$j(x, t) = \sum_{\alpha} e_{\alpha} n_{\alpha}(x, t)\, u^{\alpha}(x, t). \tag{2.15}$$

We now come to a subtler point. In order to complete the list of hydrodynamical and electrodynamical quantities of the plasma, we need to add the total energy density $\rho(x, t)\, \mathscr{E}(x, t)$, which is simply obtained from (2.4),

$$\rho(x, t)\, \mathscr{E}(x, t) = \tfrac{1}{2}\rho(x, t)\, u^2(x, t) + \rho(x, t)\, \varepsilon(x, t). \tag{2.16}$$

Because of the nonlinear nature of this definition, the relation between the global internal energy $\varepsilon(x, t)$ and the separate internal energies of the components $\varepsilon_{\alpha}(x, t)$ [or the temperatures $T_{\alpha}(x, t)$] is not simple. The reason of the difficulty is that in the two cases, the reference velocity u_R, with respect to which one defines the "random component" of the velocity $v - u_R$ is different: u_R is the centre-of-mass velocity in $\varepsilon(x, t)$, whereas it is the average velocity of the species α in $\varepsilon_{\alpha}(x, t)$.

* In the remainder of this book, the short form "electric current" will be used as a synonym for the "electric current density", whenever no confusion is possible.

In a multicomponent fluid, the total energy density $\rho(\boldsymbol{x}, t)\, \mathscr{E}(\boldsymbol{x}, t)$ [and therefore $\varepsilon(\boldsymbol{x}, t)$] is the only hydrodynamic quantity of interest, because it is a conserved quantity. This is not the case for the individual energies $\varepsilon_\alpha(\boldsymbol{x}, t)$, because the energy can be transferred from one component to another through the collisions between the molecules of unlike species (see section 2.7). But in an electron–ion plasma, we meet with a quite peculiar situation, due to the extreme smallness of the mass ratio μ. It will be shown in chapter 4 that the rate of collisional transfer of energy between ions and electrons is extremely slow. As a result, the individual component energies ε_α appear as *quasi-conserved quantities*. The plasma tends to relax towards a *quasi-equilibrium* (in a time of order τ_α) in which the temperatures of the two species are different. Subsequently, this quasi-equilibrium evolves towards a true equilibrium state in which both components have the same temperature. This process takes such a long time (of order $\mu^{-1}\tau_\alpha$) that for most purposes the quasi-equilibrium state can be considered as a quasi-stationary state.

As a result of this discussion, in an electron–ion plasma there are *two* (rather than one) (quasi-)conserved energies. It is therefore natural to consider them, or equivalently, to choose the two temperatures $T_e(\boldsymbol{x}, t)$ and $T_i(\boldsymbol{x}, t)$ defined by (2.11), along with the mass density $\rho(\boldsymbol{x}, t)$ and the barycentric velocity $\boldsymbol{u}(\boldsymbol{x}, t)$, as the set of variables necessary for a full hydrodynamic description of the plasma.

In order to complete this long, but indispensable list of definitions, we introduce the most important non-hydrodynamical quantities entering the macroscopic balance equations; their form will be clearly justified in section 3.4.

The *thermal momentum flux density*, more commonly called the *total pressure tensor* of the particles of species α is defined as

$$P_{rs}^{\alpha}(\boldsymbol{x}, t) = m_\alpha \int \mathrm{d}\boldsymbol{v}\, \left[v_r - u_r^{\alpha}(\boldsymbol{x}, t)\right]\left[v_s - u_s^{\alpha}(\boldsymbol{x}, t)\right] f^{\alpha}(\boldsymbol{v}; \boldsymbol{x}, t). \quad (2.17)$$

This quantity is, clearly, a symmetric tensor of rank two. Such a tensor can always be split into two invariant parts: one term proportional to the unit tensor δ_{rs}, and one term which has zero trace (shortly called a *traceless tensor*),

$$P_{rs}^{\alpha}(\boldsymbol{x}, t) = P_\alpha(\boldsymbol{x}, t)\delta_{rs} + \pi_{rs}^{\alpha}(\boldsymbol{x}, t), \quad (2.18)$$

where

$$P_\alpha(\boldsymbol{x}, t) = \tfrac{1}{3}\,\mathrm{Tr}\,\boldsymbol{P}^{\alpha}(\boldsymbol{x}, t) \equiv \tfrac{1}{3}P_{nn}^{\alpha}(\boldsymbol{x}, t). \quad (2.19)$$

Clearly, $P_\alpha(\boldsymbol{x}, t)$ coincides with the scalar pressure defined in (2.8). The

traceless part $\pi_{rs}^{\alpha}(x, t)$ is called the *dissipative pressure tensor* of species α. * Whenever no confusion is possible, we shall omit the word "dissipative" for brevity.

The *heat flux* (density) of species α is a vector whose components are defined as

$$q_r^{\alpha}(x, t) = \tfrac{1}{2} m_{\alpha} \int \mathrm{d}v \, [v_r - u_r^{\alpha}(x, t)] \, | v - u^{\alpha}(x, t) |^2 \, f^{\alpha}(v; x, t). \tag{2.20}$$

The heat flux, together with the particle flux, is an essential "figure-of-merit" of any plasma confinement system.

For the reader's convenience, we have collected all the basic definitions of the hydrodynamic and non-hydrodynamic variables in table 2.1, following a natural classification. (This table also contains the definition of a few moments that were not discussed in the main text, and which will be useful in the forthcoming chapters; the notations $h_{\ldots}^{\alpha(m)}$ will be introduced in section **4**.3.) The relations between one-fluid and two-fluid variables are collected in table 2.2. The classification of these quantities will be further discussed and refined in section **4**.4.

Let us note that the definitions chosen here for the various moments describing a plasma are adopted by practically all plasma physicists. However, they do *not* coincide with the definitions used in the kinetic theory of neutral gas mixtures (see, e.g. Chapman and Cowling 1952, de Groot and Mazur 1984). The main difference is in the choice of the macroscopic "reference velocity" (see the discussion on the energy density, above) in these definitions. For instance, the "traditional" definition of the pressure tensor of species α is

$$P_{rs}'^{\alpha}(x, t) = m_{\alpha} \int \mathrm{d}v \, [v_r - u_r(x, t)] [v_s - u_s(x, t)] \, f^{\alpha}(v; x, t). \tag{2.21}$$

Here, both the electron and the ion pressure tensors are referred to the same velocity, the *centre-of-mass velocity* u, instead of the separate average velocities u^{α} of each species, as in (2.17). Similar comments apply to the heat fluxes q^{α}. As a result, the equations of evolution of the moments (section 3.4) will have different forms in the two formalisms. Of course, whatever choice is made, the equations of evolution are strictly equivalent, in spite of their

* Let us note that the identification of P_{α} with the scalar pressure is only true when the *bulk viscosity* of the plasma is negligible; otherwise, the trace of the dissipative pressure tensor does not vanish. It turns out that the bulk viscosity is, indeed, negligible when the condition of weak coupling (**2**.4.10) is satisfied.

different forms. One can establish a "dictionary" enabling one to pass from one system of definitions to another. This was done extensively by Misguich and Balescu (1984).

Table 2.1
Kinetic expressions of the main quantities of macroscopic plasma physics

Quantity	Expression
PLASMADYNAMICAL ("TWO-FLUID") VARIABLES	
Number density of species α, n_α	$n_\alpha(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
Average velocity of species α, u_r^α	$n_\alpha(\boldsymbol{x}, t) u_r^\alpha(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{v} v_r f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
Temperature of species α, T_α	$n_\alpha(\boldsymbol{x}, t) T_\alpha(\boldsymbol{x}, t) = \frac{1}{3} m_\alpha \int \mathrm{d}\boldsymbol{v} \lvert \boldsymbol{v} - \boldsymbol{u}^\alpha \rvert^2 f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
HYDRODYNAMICAL ("ONE-FLUID") VARIABLES	
Mass density, ρ	$\rho(\boldsymbol{x}, t) = \sum_\alpha m_\alpha \int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
Centre-of-mass velocity, u_r	$\rho(\boldsymbol{x}, t)\, u_r(\boldsymbol{x}, t) = \sum_\alpha m_\alpha \int \mathrm{d}\boldsymbol{v} v_r f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
Temperature of species α, T_α	$n_\alpha(\boldsymbol{x}, t)\, T_\alpha(\boldsymbol{x}, t) = \frac{1}{3} m_\alpha \int \mathrm{d}\boldsymbol{v} \lvert \boldsymbol{v} - \boldsymbol{u}^\alpha \rvert^2 f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
NON-HYDRODYNAMICAL VARIABLES	
Electrodynamical variables	
Electric charge density, σ	$\sigma(\boldsymbol{x}, t) \equiv n_e h^{(0)}(\boldsymbol{x}, t) = \sum_\alpha e_\alpha \int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
Electric current density, j_r	$j_r(\boldsymbol{x}, t) \equiv e n_e (T_e/m_e)^{1/2} h_r^{(1)}(\boldsymbol{x}, t) = \sum_\alpha e_\alpha \int \mathrm{d}\boldsymbol{v} v_r f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
Fluxes	
Particle flux of species α, Γ_r^α	$\Gamma_r^\alpha(\boldsymbol{x}, t) \equiv n_\alpha (T_\alpha/m_\alpha)^{1/2} h_r^{\alpha(1)}(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{v} v_r f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
Dissipative pressure tensor	$\pi_{rs}^\alpha(\boldsymbol{x}, t) = \sqrt{2}\, n_\alpha T_\alpha\, h_{rs}^{\alpha(2)}(\boldsymbol{x}, t)$
of species α, π_{rs}^α	$= m_\alpha \int \mathrm{d}\boldsymbol{v} [(v_r - u_r^\alpha)(v_s - u_s^\alpha) - \frac{1}{3} \lvert \boldsymbol{v} - \boldsymbol{u}^\alpha \rvert^2 \delta_{rs}] f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$
Heat flux of species α, q_r^α	$q_r^\alpha(\boldsymbol{x}, t) \equiv \sqrt{\tfrac{5}{2}}\, m_\alpha n_\alpha (T_\alpha/m_\alpha)^{3/2}\, h_r^{\alpha(3)}(\boldsymbol{x}, t)$
	$= \frac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v} (v_r - u_r^\alpha) \lvert \boldsymbol{v} - \boldsymbol{u}^\alpha \rvert^2 f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$

Table 2.1 (continued)

Quantity	Expression
Miscellaneous moments of interest	
Total pressure tensor of species α, P^α_{rs}	$P^\alpha_{rs} = m_\alpha \int \mathrm{d}\boldsymbol{v}\,(v_r - u^\alpha_r)(v_s - u^\alpha_s) f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t)$
Third-rank tensorial third moment of species α, q^α_{rsp}	$q^\alpha_{rsp} = \frac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}\,(v_r - u^\alpha_r)(v_s - u^\alpha_s)(v_p - u^\alpha_p) f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t)$
Second-rank tensorial fourth moment of species α, S^α_{rs}	$S^\alpha_{rs} = \frac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}\,(v_r - u^\alpha_r)(v_s - u^\alpha_s) \lvert \boldsymbol{v} - \boldsymbol{u}^\alpha \rvert^2 f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t)$
Fourth-rank tensorial fourth moment of species α, S^α_{rspq}	$S^\alpha_{rspq} = \frac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}\,(v_r - u^\alpha_r)(v_s - u^\alpha_s)(v_p - u^\alpha_p)(v_q - u^\alpha_q) \times f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t)$
Third-rank tensorial fifth moment of species α, T^α_{rsp}	$T^\alpha_{rsp} = \frac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}\,(v_r - u^\alpha_r)(v_s - u^\alpha_s)(v_p - u^\alpha_p) \lvert \boldsymbol{v} - \boldsymbol{u}^\alpha \rvert^2 \times f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t)$

Table 2.2
One-fluid and two-fluid variables.

I. One-fluid variables expressed in terms of two-fluid variables

$$\rho = m_e n_e + m_i n_i \qquad u_r = \frac{m_e n_e u^e_r + m_i n_i u^i_r}{m_e n_e + m_i n_i}$$

$$\sigma = -en_e + Zen_i \qquad j_r = -en_e u^e_r + Zen_i u^i_r$$

II. Two-fluid variables expressed in terms of one-fluid variables

	A	B	C
n_e	$\dfrac{Ze\rho - m_i\sigma}{e(m_i + Zm_e)}$	$\dfrac{Z}{m_i}\rho - \dfrac{1}{e}\sigma$	$\dfrac{Z}{m_i}\rho$
n_i	$\dfrac{e\rho + m_e\sigma}{e(m_i + Zm_e)}$	$\dfrac{1}{m_i}\rho$	$\dfrac{1}{m_i}\rho$
u^e_r	$\dfrac{Ze\rho u_r - m_i j_r}{Ze\rho - m_i\sigma}$		$u_r - \dfrac{m_i}{Ze\rho} j_r$
u^i_r	$\dfrac{e\rho u_r + m_e j_r}{e\rho + m_e\sigma}$	u_r	u_r

A. Exact expressions. B. Expressions in which terms of order μ are neglected. C. Expressions valid when local electroneutrality is assumed (see section 4.1).

Here we adopt, once for all, the set of definitions given in table 2.1. The reason lies rather deeply in the picture of a plasma as a system of *two fluids* which, because of the smallness of the mass ratio μ, are almost independent. This feature makes the hydrodynamics of an electron–ion plasma very different from, say, a mixture of oxygen and nitrogen, where the component molecules have comparable masses.

3.3. Kinetic equation revisited

We have seen in section 3.2 that whenever a truly macroscopic description of a plasma is valid, the state of the plasma should be completely specified in terms of the hydrodynamical variables $\rho(\boldsymbol{x}, t)$, $\boldsymbol{u}(\boldsymbol{x}, t)$, $T_e(\boldsymbol{x}, t)$, $T_i(\boldsymbol{x}, t)$ and of the electrodynamical variables $\sigma(\boldsymbol{x}, t)$ and $\boldsymbol{j}(\boldsymbol{x}, t)$. The evolution of the plasma at this level would then be described by a set of hydrodynamical equations for the former variables, coupled to the Maxwell equations, which involve the latter variables as source terms for the electromagnetic fields. Clearly, in a kinetic theory, these quantities cannot be introduced arbitrarily. Rather, the macroscopic equations of evolution are induced by the fundamental microscopic laws of molecular motion. In other words, the hydrodynamic and electrodynamic equations must be *derived* from the kinetic equation.

We reconsider here the kinetic equation, by collecting the results obtained in chapter 2. We slightly transform the latter by writing an equation for the *local distribution functions* $f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)$, eq. (1.10). To this purpose, we multiply both sides of eq. (2.4.15) by $\delta(\boldsymbol{q}_1 - \boldsymbol{x})$ and integrate over $\boldsymbol{q}_1$, a rather trivial operation. The final result is obtained by using eqs. (2.2.21), (2.5.9), (2.5.12) and (2.6.24).

The kinetic equation expresses the rate of change of the distribution functions as a result of three causes: the free flow of the particles, the action of an electric and a magnetic field, and the action of the collisions. It can be written schematically as

$$\partial_t f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t) = \Phi^\alpha + \mathscr{F}^\alpha + \mathscr{K}^\alpha. \tag{3.1}$$

The *free flow term* Φ^α is the simplest of all,

$$\Phi^\alpha = -\boldsymbol{v} \cdot \nabla f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t). \tag{3.2}$$

This is a *linear* operator acting on the distribution function. Its most important feature (from the standpoint of transport theory) is the presence of the factor $\boldsymbol{v}$. As will be seen in the next section, it is precisely this feature which causes the macroscopic hydrodynamical equations to have a "hierarchical" structure. It is at the origin of the difficulties of the transport problem.

The *electromagnetic field term* $\mathscr{F}^\alpha$ also has a simple form, derivable from a Lie bracket,

$$\mathscr{F}^\alpha = -\frac{e_\alpha}{m_\alpha}\left(E_r(x, t) + \frac{1}{c}\varepsilon_{rmn}v_m B_n(x, t)\right)\frac{\partial}{\partial v_r}f^\alpha(v; x, t). \tag{3.3}$$

We recognize here the *Lorentz force* produced by the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$. (Remember that we use Gaussian units, in which $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same dimensions). We recall that $\boldsymbol{E}$ and $\boldsymbol{B}$ are *self-consistent fields*, determined by the distribution functions; hence $\mathscr{F}^\alpha$ is a *non-linear* term (see section 2.5).

The plasma acts like a dielectric and magnetic medium: the action of an electromagnetic field produces a polarization and a magnetization which, in turn, modify the initial field. In macroscopic electrodynamics one is forced to introduce phenomenological assumptions in order to relate the polarization and the magnetization to the external field, and therefore to close the Maxwell equations. In kinetic theory we do not need such additional assumptions. Indeed, we note that the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ of eq. (3.3) are determined by the Maxwell equations in which the source terms, i.e. the charge density σ and the current density $\boldsymbol{j}$ are defined in terms of the distribution functions through (2.14) and (2.15). Thus, the kinetic equation (3.1) must be completed by the Maxwell equations,

$$\nabla \cdot \boldsymbol{E}(x, t) = 4\pi \sum_\alpha e_\alpha \int \mathrm{d}v\, f^\alpha(v; x, t), \tag{3.4}$$

$$\nabla \wedge \boldsymbol{E}(x, t) = -\frac{1}{c}\partial_t \boldsymbol{B}(x, t), \tag{3.5}$$

$$\nabla \cdot \boldsymbol{B}(x, t) = 0, \tag{3.6}$$

$$\nabla \wedge \boldsymbol{B}(x, t) = \frac{1}{c}\partial_t \boldsymbol{E}(x, t) + \frac{4\pi}{c}\sum_\alpha e_\alpha \int \mathrm{d}v\, v f^\alpha(v; x, t). \tag{3.7}$$

The collision term $\mathscr{K}^\alpha$ is by far the most complicated, but also one of the most important drivers of the evolution. It represents the rate of change of the distribution functions due to the elastic "collision" processes. Such a process can be defined as an event involving a small group of particles in a *region of finite size* – say, a sphere of radius r_c, the range of the interactions – in which the particles interact for a *finite time* – say, τ_c, the duration of a collision.

The motion of billiard balls provides a classical picture of the motion driven by two-body collisions. In this example, it is worth thinking of an

aspect which is rarely discussed in textbooks. The motion of two billiard balls before and after their mutual collision is *not* a free motion. Indeed, in this interval they are acted upon by the *external gravitational field*, as well as by the presence of a *hard table with vertical walls*. Hence, the motion of the billiard balls between two collisions is *constrained* by the presence of external forces. The effect of the constraint is to confine the motion to a plane limited by a rectangular boundary.

Analogously, the motion of the electrons and ions in a plasma between two collisions is not a free three-dimensional motion, but rather a motion determined by the presence of an electromagnetic field acting continuously on the particles. There is, however, a difference with the example of the billiard balls. The constraint in the plasma is due not only to an external field, but also to the average electromagnetic field produced by the particles themselves. In other words, the effect of the long-range interactions in a plasma is twofold: there is a *persistent* action through the Vlasov field, as well as an *intermittent* action through localized collisions of small groups of particles.

We have shown in chapter 2 that in the simple, but important case of a weakly coupled plasma, the collisions are described by the Landau collision operator (2.6.24) which, for local distribution functions, becomes

$$\mathscr{K}^{\alpha} = \sum_{\beta} \mathscr{K}^{\alpha\beta},$$

with

$$\mathscr{K}^{\alpha\beta} = 2\pi e_{\alpha}^{2} e_{\beta}^{2} \ln \Lambda \int \mathrm{d}\boldsymbol{v}_2 \; m_{\alpha}^{-1} \frac{\partial}{\partial v_{1m}} G_{mn}(\boldsymbol{g})$$

$$\times \left(m_{\alpha}^{-1} \frac{\partial}{\partial v_{1n}} - m_{\beta}^{-1} \frac{\partial}{\partial v_{2n}} \right) f^{\alpha}(\boldsymbol{v}_1;\, \boldsymbol{x},\, t)\, f^{\beta}(\boldsymbol{v}_2;\, \boldsymbol{x},\, t), \tag{3.8}$$

where the Landau tensor is defined as

$$G_{mn}(\boldsymbol{g}) = \frac{g^2 \delta_{mn} - g_m g_n}{g^3}. \tag{3.9}$$

3.4. *Equations of evolution of the macroscopic quantities*

Having summarized the results of chapter 2, we proceed to the key step which consists of deriving the laws of macroscopic evolution from the microscopic

dynamical laws. This matter is treated in most textbooks on plasma physics, such as those quoted in section 3.2. An often quoted reference is the work by Braginskii (1965); a short but very clear treatment is given in the article by Freidberg (1982). In most of these works, the subject is, however, not treated exhaustively.

The derivation is most conveniently done in two steps. We first derive *equations of evolution for the two-field (plasmadynamical) variables* n_α, u_r^α, T_α. The definitions in table 2.1 immediately suggest a general procedure for this derivation. We multiply both sides of the kinetic equation (3.1) successively by 1, v_r, $|\boldsymbol{v}-\boldsymbol{u}^\alpha|^2$ and integrate over the velocity $\boldsymbol{v}$. The right-hand sides of the resulting equations must be analyzed in order to be expressed in terms of macroscopic quantities.

Integrating directly (3.1) over the velocities, we obtain

$$\partial_t \int \mathrm{d}\boldsymbol{v}\, f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t) \equiv \partial_t n_\alpha(\boldsymbol{x},\, t) = \int \mathrm{d}\boldsymbol{v}\, (\Phi^\alpha + \mathscr{F}^\alpha + \mathscr{K}^\alpha), \tag{4.1}$$

where we used definition (2.2). Using now (3.2) we have

$$\int \mathrm{d}\boldsymbol{v}\, \Phi^\alpha = -\int \mathrm{d}\boldsymbol{v}\, v_m \nabla_m f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t) = -\nabla_m \int \mathrm{d}\boldsymbol{v}\, v_m f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t)$$

$$= -\nabla_m \left[n_\alpha(\boldsymbol{x},\, t)\, u_m^\alpha(\boldsymbol{x},\, t) \right], \tag{4.2}$$

where we used eq. (2.3). The Vlasov term is written explicitly as (remember the abbreviation: $\partial_r \equiv \partial/\partial v_r$)

$$\int \mathrm{d}\boldsymbol{v}\, \mathscr{F}^\alpha = -\frac{e_\alpha}{m_\alpha} \int \mathrm{d}\boldsymbol{v} \left(E_r(\boldsymbol{x},\, t) + \frac{1}{c} \varepsilon_{rmn} v_m B_n(\boldsymbol{x},\, t) \right) \partial_r f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t). \tag{4.3}$$

In order to deal with this term, we use the two lemmas (**2**.7.1), (**2**.7.2) of chapter 2,

$$\int \mathrm{d}\boldsymbol{v} \left(E_r(\boldsymbol{x},\, t) + c^{-1} \varepsilon_{rmn} v_m B_n(\boldsymbol{x},\, t) \right) \partial_r f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t)$$

$$= c^{-1} \varepsilon_{rmn} \delta_{rm} B_n(\boldsymbol{x},\, t) \int \mathrm{d}\boldsymbol{v}\, f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t) \equiv 0, \tag{4.4}$$

the vanishing being due to the antisymmetry of the Levi-Civita symbol.

Finally, we note that the general conservation law (2.7.4) ensures the vanishing of the last term in (4.1). Collecting all these results, we finally obtain *

$$\partial_t n_\alpha = -\nabla \cdot (n_\alpha \boldsymbol{u}^\alpha), \quad \alpha = \text{e,i}. \tag{4.5}$$

We now multiply both sides of (3.1) by $m_\alpha v_r$ and integrate over $\boldsymbol{v}$. Using (2.3), we obtain

$$m_\alpha \, \partial_t (n_\alpha u_r^\alpha) = m_\alpha \int \mathrm{d}\boldsymbol{v} \, v_r (\Phi^\alpha + \mathscr{F}^\alpha + \mathscr{K}^\alpha). \tag{4.6}$$

Proceeding as above, we find

$$\begin{aligned} m_\alpha \int \mathrm{d}\boldsymbol{v} \, v_r \, \Phi^\alpha &= -\nabla_m m_\alpha \int \mathrm{d}\boldsymbol{v} \, v_r \, v_m f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t) \\ &= -\nabla_m m_\alpha \Big[u_r^\alpha u_m^\alpha \int \mathrm{d}\boldsymbol{v} \, f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t) \\ &\qquad + \int \mathrm{d}\boldsymbol{v} \, (v_r - u_r^\alpha)(v_m - u_m^\alpha) f^\alpha(\boldsymbol{v};\, \boldsymbol{x},\, t) \Big] \\ &= -\nabla_m \big[m_\alpha n_\alpha u_r^\alpha u_m^\alpha + n_\alpha T_\alpha \, \delta_{rm} + \pi_{rm}^\alpha \big]. \end{aligned} \tag{4.7}$$

In going from the second to the third form we have simply written $v_r \, v_m$ as $(v_r - u_r^\alpha + u_r^\alpha)\,(v_m - u_m^\alpha + u_m^\alpha)$; in going to the last form, we used definitions (2.18), (2.19) and (2.11). A simple integration by parts provides us with the result

$$m_\alpha \int \mathrm{d}\boldsymbol{v} \, v_r \mathscr{F}^\alpha = e_\alpha n_\alpha \big(E_r + c^{-1} \varepsilon_{rmn} u_m^\alpha B_n \big). \tag{4.8}$$

Finally, we introduce the important vector quantity

$$R_r^\alpha(\boldsymbol{x},\, t) \equiv m_\alpha \int \mathrm{d}\boldsymbol{v} \, v_r \, \mathscr{K}^\alpha. \tag{4.9}$$

It is called the *friction force* density acting on the particles of species α. Collecting the results, we obtain

$$\begin{aligned} \partial_t (m_\alpha n_\alpha u_r^\alpha) = &-\nabla_m (m_\alpha n_\alpha u_r^\alpha u_m^\alpha + \delta_{rm} n_\alpha T_\alpha + \pi_{rm}^\alpha) \\ &+ e_\alpha n_\alpha \big(E_r + c^{-1} \varepsilon_{rmn} u_m^\alpha B_n \big) + R_r^\alpha. \end{aligned} \tag{4.10}$$

* From here on we shall very often omit writing explicitly the arguments $(\boldsymbol{x}, t)$ of the macroscopic field variables.

The equations for the *temperatures* are obtained by multiplying both sides of (3.1) by $\frac{1}{3}m_\alpha |\boldsymbol{v}-\boldsymbol{u}^\alpha|^2$, integrating over $\boldsymbol{v}$ and analyzing the terms as before. The result is

$$n_\alpha \partial_t T_\alpha = -n_\alpha \boldsymbol{u}^\alpha \cdot \nabla T_\alpha - \tfrac{2}{3} n_\alpha T_\alpha \nabla \cdot \boldsymbol{u}^\alpha - \tfrac{2}{3}\pi^\alpha_{mn} \nabla_m u^\alpha_n - \tfrac{2}{3}\nabla_m q^\alpha_m + \tfrac{2}{3} Q^\alpha, \tag{4.11}$$

where the *collisional rate of heat exchange* is defined as

$$Q^\alpha = \tfrac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}\, |\boldsymbol{v}-\boldsymbol{u}^\alpha|^2 \mathscr{K}^\alpha. \tag{4.12}$$

The quantities $\boldsymbol{R}^\alpha$ and Q^α are not independent. The conservation laws (2.7.4)–(2.7.6) imply some important relations among them. We first note that, because of (2.7.9)–(2.7.11), these quantities are solely determined by the *unlike-particle collisions* $\mathscr{K}^{\alpha\beta}$ $(\alpha \neq \beta)$. They may therefore naturally be denoted by quantities with two superscripts:

$$\boldsymbol{R}^{\mathrm{e}} \equiv \boldsymbol{R}^{\mathrm{ei}}, \qquad \boldsymbol{R}^{\mathrm{i}} \equiv \boldsymbol{R}^{\mathrm{ie}},$$

$$Q^{\mathrm{e}} \equiv Q^{\mathrm{ei}}, \qquad Q^{\mathrm{i}} \equiv Q^{\mathrm{ie}}. \tag{4.13}$$

Next, eqs. (2.7.12)–(2.7.14) express the relations

$$\boldsymbol{R}^{\mathrm{ei}} = -\boldsymbol{R}^{\mathrm{ie}}, \tag{4.14}$$

$$Q^{\mathrm{ei}} = -Q^{\mathrm{ie}} - (\boldsymbol{u}^{\mathrm{e}} - \boldsymbol{u}^{\mathrm{i}}) \cdot \boldsymbol{R}^{\mathrm{ei}}. \tag{4.15}$$

Equations (4.5), (4.10) and (4.11) will be called the *plasmadynamical balance equations*. Another current name is: the *two-fluid balance equations* *.

This name is rather natural for the following reason. If $\boldsymbol{R}^\alpha$ and Q^α were negligible, these equations would simply be the hydrodynamical balance equations for two *separate, independent fluids*: the electrons and the ions. Equations (4.5) are the continuity equations. Equations (4.10) are the momentum balance equations, expressing the change of the momentum density as a result of the divergence of the convective momentum flux and of the total pressure tensor, as well as of the action of the Lorentz force on the charged particles. Finally, the energy balance equations (4.11) express the rate of change of the temperatures as due to an inertial term, to the rate of work done

* Very often, in works on plasma physics, only eqs. (4.5) and (4.10) are considered under the name of two-fluid equations.

by the pressure forces and to the divergence of the heat flux $\boldsymbol{q}^\alpha$. But actually, the two fluids are *not* independent. This is due to the fact that the electron and ion momentum and energy are not separately conserved. The terms $\boldsymbol{R}^\alpha$ and Q^α precisely take into account the transfer of momentum and of energy between the two fluids.

The two fluids are also coupled in another way. Indeed, the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in (4.10) are determined by the Maxwell equations (3.4)–(3.7) which are written in terms of the plasma-dynamical variables as

$$\nabla \cdot \boldsymbol{E} = 4\pi \sum_\alpha e_\alpha n_\alpha, \tag{4.16}$$

$$\nabla \wedge \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B}, \tag{4.17}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{4.18}$$

$$\nabla \wedge \boldsymbol{B} = \frac{1}{c}\partial_t \boldsymbol{E} + \frac{4\pi}{c}\sum_\alpha e_\alpha n_\alpha \boldsymbol{u}^\alpha. \tag{4.19}$$

Thus, the electromagnetic fields are determined by both the electron and the ion variables.

The two-fluid picture of the plasma is often convenient. However, in many cases it is important to stress the behaviour of the true hydrodynamical variables of the plasma. We therefore transform the preceding equations and derive the *hydrodynamical balance equations* (or *one-fluid balance equations*).

We linearly combine the two equations (4.5) as indicated by (2.12) and obtain the *continuity equation*

$$\partial_t \rho = -\nabla \cdot (\rho \boldsymbol{u}). \tag{4.20}$$

Using the transformation formulae of table 2.2, we obtain the *momentum balance equation* by combining the two equations (4.10). Its exact form is rather complicated, but it can be considerably simplified by taking into account the relation $\mu \ll 1$, and neglecting terms of order $\mu (\equiv m_e/m_i)$. We then obtain

$$\partial_t \rho u_r = -\nabla_s \left\{ \rho u_r u_s + \delta_{rs} \left[\left(\frac{Z\rho}{m_i} - \frac{\sigma}{e} \right) T_e + \frac{\rho}{m_i} T_i \right] + \pi_{rs}^e + \pi_{rs}^i \right\}$$
$$+ \sigma E_r + c^{-1} \varepsilon_{rsm} j_s B_m. \tag{4.21}$$

This equation has a very simple interpretation in terms of hydrodynamical concepts. The right-hand side contains the divergence of a sum of three terms:

the convective momentum flux, the scalar pressure and the dissipative pressure tensor. It shows, in particular, that the expression of the total scalar pressure, i.e. the equation of state, is

$$P = \left(\frac{Z\rho}{m_i} - \frac{\sigma}{e} \right) T_e + \frac{\rho}{m_i} T_i. \tag{4.22}$$

Besides the divergence, the right-hand side contains the Lorentz force density, coupling the equation of motion to the Maxwell equations, which are now written as

$$\nabla \cdot \boldsymbol{E} = 4\pi\sigma, \tag{4.23}$$

$$\nabla \wedge \boldsymbol{E} = -\frac{1}{c} \partial_t \boldsymbol{B}, \tag{4.24}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{4.25}$$

$$\nabla \wedge \boldsymbol{B} = \frac{1}{c} \partial_t \boldsymbol{E} + \frac{4\pi}{c} \boldsymbol{j}. \tag{4.26}$$

It is important to note that there is no contribution of the collision term $\mathscr{X}^\alpha$ to (4.18). Indeed, the two terms $\boldsymbol{R}^e$ and $\boldsymbol{R}^i$ of eq. (4.10) cancel exactly as a result of (4.14). This expresses the local conservation of the total momentum by the collisions.

We now turn to the equations for the *temperatures*. As explained in section 3.2, we shall keep the two separate temperatures as hydrodynamical variables, *even in the one-fluid picture*, because the individual internal energies are quasi-conserved quantities (as will be shown explicitly at the end of section 4.6). We therefore rewrite eqs. (4.11), neglecting terms of order μ, as

$$\begin{aligned} \left(\frac{Z\rho}{m_i} - \frac{\sigma}{e} \right) \partial_t T_e = &- \left(\frac{Z\rho}{m_i} u_r - \frac{1}{e} j_r \right) \nabla_r T_e \\ &- \tfrac{2}{3} \left(\frac{Z\rho}{m_i} - \frac{\sigma}{e} \right) T_e \nabla_r \left(\frac{Ze\rho u_r - m_i j_r}{Ze\rho - m_i \sigma} \right) \\ &- \tfrac{2}{3} \pi^e_{rs} \nabla_r \left(\frac{Ze\rho u_s - m_i j_s}{Ze\rho - m_i \sigma} \right) - \tfrac{2}{3} \nabla_r q^e_r + \tfrac{2}{3} Q^e, \end{aligned} \tag{4.27}$$

$$\frac{\rho}{m_i} \partial_t T_i = -\frac{\rho}{m_i} u_r \nabla_r T_i - \tfrac{2}{3} \frac{\rho}{m_i} T_i \nabla_r u_r - \tfrac{2}{3} \pi^i_{rs} \nabla_r u_s - \tfrac{2}{3} \nabla_r q^i_r + \tfrac{2}{3} Q^i. \tag{4.28}$$

A characteristic feature of the plasma is the fact that the balance equations (4.21), (4.27) and (4.28) involve the *electrodynamical variables* σ and $\boldsymbol{j}$ along with the hydrodynamical variables ρ, $\boldsymbol{u}$, T_e, T_i. We must therefore complete the set of macroscopic equations by adding two more equations for the former.

The equation for the *charge density* is immediately obtained, without any approximation, from (4.5):

$$\partial_t \sigma = -\nabla_r j_r. \tag{4.29}$$

This is the well-known balance equation expressing the *conservation of the electrical charge*; it is, of course, consistent with the Maxwell equations (4.23)–(4.26) and is actually superfluous.

The equation for the current, derived from (4.10) with the neglect of terms of order μ, is much more complicated *:

$$\begin{aligned}\partial_t j_r = \nabla_s \left(\frac{Ze\rho\sigma u_r u_s - Ze\rho(j_r u_s + j_s u_r) + m_i j_r j_s}{Ze\rho - m_i\sigma} \right) \\ + \frac{1}{m_e m_i} \nabla_r \left[(Ze\rho - m_i\sigma) T_e \right] + \frac{e}{m_e} \nabla_s \left(\pi_{rs}^e - Z\mu\pi_{rs}^i \right) \\ + \frac{e}{m_e m_i} (Ze\rho - m_i\sigma) E_r + \frac{e}{c m_e m_i} \varepsilon_{rsm} (Ze\rho u_s - m_i j_s) B_m \\ - \frac{e}{m_e} R_r^e.\end{aligned} \tag{4.30}$$

This equation is called the *generalized Ohm law*. It plays the same role as the ordinary Ohm law, in providing a relation between the current $\boldsymbol{j}$ and the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, as well as the hydrodynamical variables ρ, $\boldsymbol{u}$, T_e, T_i. The fact that the latter variables enter the expression of the electrodynamical quantities is another characteristic feature of the plasma, as compared to systems studied in elementary electrodynamics. We will see in chapter 5 under which conditions eq. (4.30) reduces to the usual Ohm law.

An additional balance equation which, in a sense, supervises all the others, is the *entropy balance equation*: it will be derived in section 3.5.

We have now completed the derivation of the basic equations of macroscopic plasma physics. They are the *hydrodynamical balance equations* (4.20),

* Note that we have not neglected the term $\mu\pi_{rs}^i$, because we do not know *a priori* its size, compared to π_{rs}^e.

(4.21), (4.27), (4.28), the *generalized Ohm law* (4.30) and the *Maxwell equations* (4.23)–(4.26).

This is, of course, only the beginning of our task! It is clearly apparent that *these equations are not closed*. They involve, besides the hydrodynamical and the electrodynamical variables, the additional unknowns

$$\pi^{e}_{rs},\ \pi^{i}_{rs},\ q^{e}_{r},\ q^{i}_{r},\ R^{e}_{r},\ Q^{e},\ Q^{i}. \tag{4.31}$$

The hydrodynamical equations have a typical *hierarchical structure*. The equations for the first moments ($\boldsymbol{u}$, $\boldsymbol{j}$) involve the second moments ($\boldsymbol{\pi}^{e}$, $\boldsymbol{\pi}^{i}$), and the equations for the second moments involve the third moments ($\boldsymbol{q}^{e}$, $\boldsymbol{q}^{i}$). Hence, the equations for the third moments will involve the fourth moments, and so on *ad infinitum*. The origin of this hierarchical behaviour lies in the simple-looking free flow term (3.2) of the kinetic equation, with its factor $\boldsymbol{v}$, as well as in the complicated collision term.

These features are illustrated by the following balance equations, which are easily derived from the kinetic equation by the same methods as above. These equations are not merely illustrations of the previous statements: they will be used in the forthcoming transport theory of toroidally confined plasmas. The equations concern two vectorial fluxes: the particle fluxes Γ^{α} and the heat fluxes $\boldsymbol{q}^{\alpha}$, and two tensorial fluxes: the (total) pressure tensors $\boldsymbol{P}^{\alpha}$ and the generalized fourth-order pressure tensors $\boldsymbol{S}^{\alpha}$. All these quantities are defined in table 2.1. We thus obtain

Particle flux Γ^{α}:

$$\partial_t \Gamma^{\alpha}_r = -\nabla_n\left(m_{\alpha}^{-1} P^{\alpha}_{rn} + u^{\alpha}_n \Gamma^{\alpha}_r\right) + \frac{e_{\alpha}}{m_{\alpha}} n_{\alpha} E_r + \frac{e_{\alpha} B}{m_{\alpha} c}\varepsilon_{rmn}\Gamma^{\alpha}_m b_n + \frac{1}{m_{\alpha}} R^{\alpha}_r. \tag{4.32}$$

Heat flux $\boldsymbol{q}^{\alpha}$:

$$\begin{aligned}\partial_t q^{\alpha}_r = &-\nabla_n\left(S^{\alpha}_{rn} + u^{\alpha}_n q^{\alpha}_r\right) - \left(2q^{\alpha}_{rmn} + q^{\alpha}_m \delta_{rn}\right)\nabla_m u^{\alpha}_n \\ &+ \left(P^{\alpha}_{rn} + \tfrac{3}{2} P_{\alpha}\,\delta_{rn}\right)\frac{1}{m_{\alpha} n_{\alpha}}\nabla_m P^{\alpha}_{mn} \\ &+ \frac{e_{\alpha} B}{m_{\alpha} c}\varepsilon_{rmn} q^{\alpha}_m b_n + R^{\alpha(3)}_r - \frac{1}{m_{\alpha} n_{\alpha}}\left(P^{\alpha}_{rn} + \tfrac{3}{2} P_{\alpha}\,\delta_{rn}\right) R^{\alpha}_n.\end{aligned} \tag{4.33}$$

Pressure tensor $\boldsymbol{P}^{\alpha}$:

$$\partial_t P^{\alpha}_{rs} = -\nabla_n \left(2q^{\alpha}_{rsn} + u^{\alpha}_n P^{\alpha}_{rs}\right) - \left(P^{\alpha}_{rm}\,\delta_{sn} + P^{\alpha}_{sm}\,\delta_{rn}\right)\nabla_m u^{\alpha}_n$$

$$+ \frac{e_\alpha B}{m_\alpha c}\left(\varepsilon_{rpq} P^{\alpha}_{sp} + \varepsilon_{spq} P^{\alpha}_{rp}\right) b_q + R^{\alpha(2)}_{rs}. \tag{4.34}$$

Fourth-order pressure tensor $\boldsymbol{S}^{\alpha}$:

$$\partial_t S^{\alpha}_{rs} = -\nabla_n \left(T^{\alpha}_{rsn} + u^{\alpha}_n S^{\alpha}_{rs}\right) - \left(2S^{\alpha}_{rnm} + S^{\alpha}_{rm}\,\delta_{sn} + S^{\alpha}_{sm}\,\delta_{rn}\right)\nabla_m u^{\alpha}_n$$

$$+ \left(2q^{\alpha}_{rsn} + q^{\alpha}_r \delta_{sn} + q^{\alpha}_s \delta_{rn}\right)\frac{1}{m_\alpha n_\alpha}\nabla_m P^{\alpha}_{mn}$$

$$+ \frac{e_\alpha B}{m_\alpha c}\left(\varepsilon_{rpq} S^{\alpha}_{sp} + \varepsilon_{spq} S^{\alpha}_{rp}\right) b_q$$

$$+ R^{\alpha(4)}_{rs} - \frac{1}{m_\alpha n_\alpha}\left(2q^{\alpha}_{rsn} + q^{\alpha}_r \delta_{sn} + q^{\alpha}_s \delta_{rn}\right) R^{\alpha}_n. \tag{4.35}$$

We note that these equations have a common structure. Each moment equation involves a term containing the divergence of a higher moment (as stated above), together with the moment itself, multiplied by $\boldsymbol{u}^{\alpha}$ (thus: $\partial_t \Gamma^{\alpha}$ contains $\boldsymbol{P}^{\alpha} + \boldsymbol{u}^{\alpha}\Gamma^{\alpha}$). All equations contain a term involving the magnetic field in the form of a generalized Lorentz force [thus: $\partial_t \boldsymbol{q}^{\alpha}$ contains $(e_\alpha B/m_\alpha c)$ $(\boldsymbol{q}^{\alpha} \wedge \boldsymbol{b})$]. All equations contain a contribution of the collision term (because these moments are not conserved quantities!) in the form of generalized friction forces $\boldsymbol{R}^{\alpha(n)}$,

$$R^{\alpha(3)}_r = \tfrac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}\, |\boldsymbol{v} - \boldsymbol{u}^{\alpha}|^2 \left(v_r - u^{\alpha}_r\right) \mathscr{K}^{\alpha}, \tag{4.36}$$

$$R^{\alpha(2)}_{rs} = m_\alpha \int \mathrm{d}\boldsymbol{v} \left(v_r - u^{\alpha}_r\right)\left(v_s - u^{\alpha}_s\right) \mathscr{K}^{\alpha}, \tag{4.37}$$

$$R^{\alpha(4)}_{rs} = \tfrac{1}{2} m_\alpha \int \mathrm{d}\boldsymbol{v}\, |\boldsymbol{v} - \boldsymbol{u}^{\alpha}|^2 \left(v_r - u^{\alpha}_r\right)\left(v_s - u^{\alpha}_s\right) \mathscr{K}^{\alpha}. \tag{4.38}$$

[The ordinary friction $\boldsymbol{R}^{\alpha}$ was defined in eq. (4.9).] The higher order equations contain also terms proportional to $\nabla_m u^{\alpha}_n$ and to $\nabla_m P^{\alpha}_{mn}$.

A feature worth emphasizing is the fact that *the electric field only enters the equation for the particle flux* Γ^{α}. This is due to the fact that all moments

considered here, *except* Γ^{α}, are so-called *centred moments*, i.e. averages of powers of $(\boldsymbol{v}-\boldsymbol{u}^{\alpha})$ (rather than of $\boldsymbol{v}$ alone). As a result, in deriving these equations, one finds remarkable cancellations leading to the annihilation of all the terms involving the electric field in these higher moment equations.

Examples (4.32)–(4.35) clearly show that there is no possibility of an exact closure of the system of hydrodynamical equations by simply writing more and more higher moment equations. What we need now is a deeper analysis of the underlying kinetic equation, leading to an approximate solution of the problem.

The approximate closure of the hydrodynamical equations, valid under specified conditions, is the purpose of *Transport Theory*. Its study is the subject of the remainder of this book. It will be the opportunity of an excursion through the most varied fields of physics. For this reason, apart from its great practical importance, transport theory is a treasure of beauty, still largely untouched...

3.5. *The entropy balance*

It is well known that the central role in thermodynamics is played by the concept of *Entropy*. As thermodynamics is, in turn, the most general formulation of macroscopic physics, it is clear that the entropy will play the role of a monitor for all the theories of macroscopic (hydrodynamic and electrodynamic) plasma behaviour.

The importance of the entropy comes from the celebrated *second law of thermodynamics*, which can be formulated in a very general way as follows. We consider a subdivision of the "universe" into two parts: the "*system*" and the "*external world*"; these two parts are separated by a "*boundary*" Ω (which may be conceived either as a physical wall or membrane, or as a purely fictitious mathematical closed surface) (see fig. 5.1). The volume of the system is

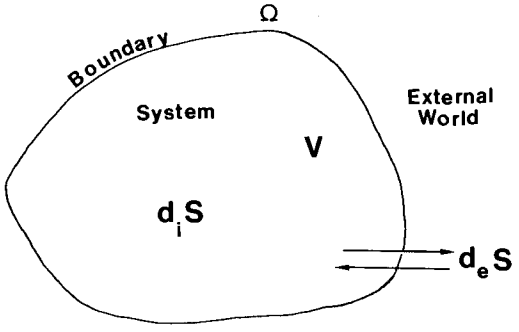

Fig. 5.1. The system and the external world. The two parts of the entropy variation.

denoted by V. The second law of thermodynamics postulates the existence of a state function, the *entropy* S. This statement implies that the value of the entropy is uniquely determined by the values of the variables defining the *macroscopic state* of the system (densities, temperatures, ...).

Whenever anything happens to the system, there results a variation of the entropy, which is denoted by $\mathrm{d}S$. It is a trivial remark to note that $\mathrm{d}S$ can always be decomposed into two terms (Prigogine and Defay 1950, de Groot and Mazur 1984),

$$\mathrm{d}S = \mathrm{d}_e S + \mathrm{d}_i S. \tag{5.1}$$

The term $\mathrm{d}_e S$ represents the amount of entropy which *enters or leaves the system* as a result of interactions with the external world; in other words, the quantity $\mathrm{d}_e S$ *crosses the boundary*. On the contrary, $\mathrm{d}_i S$ represents the entropy produced or destroyed as a result of *internal processes*, and which *remains within the system*. The non-trivial part of the second law consists of the postulate that the change of entropy due to internal processes has a well-defined sign,

$$\mathrm{d}_i S \geqslant 0. \tag{5.2}$$

This statement – the only fundamental law of physics expressed by an *inequality* rather than an equation – contains in itself all the laws determining the irreversible evolution of the macroscopic systems.

In the framework of non-equilibrium thermodynamics, the main objective is to relate the rate of change *in time* of the entropy to the various hydrodynamical, electrodynamical or chemical processes occurring in the system. In the case when the system is a certain amount of fluid, a *local* formulation of the entropy balance is preferable. We therefore define a *specific entropy* (per particle) s, a total *entropy flux* per unit area and unit time, $\boldsymbol{J}_{S\,\mathrm{tot}}$, and an *entropy source strength* σ; all these quantities are locally defined fields, i.e. functions of position $\boldsymbol{x}$ and of time t. They are related to the global quantities as

$$\begin{aligned} S &= \int_V \mathrm{d}\boldsymbol{x}\, n(\boldsymbol{x}, t)\, s(\boldsymbol{x}, t), \\ \frac{\mathrm{d}_e S}{\mathrm{d}t} &= -\int_\Omega \mathrm{d}\boldsymbol{A} \cdot \boldsymbol{J}_{S\,\mathrm{tot}}(\boldsymbol{x}, t), \\ \frac{\mathrm{d}_i S}{\mathrm{d}t} &= \int_V \mathrm{d}\boldsymbol{x}\, \sigma(\boldsymbol{x}, t). \end{aligned} \tag{5.3}$$

The global entropy balance is then transformed into a *local entropy balance*,

$$\partial_t(ns) = -\nabla \cdot \boldsymbol{J}_{S\text{tot}} + \sigma. \tag{5.4}$$

The main point which remains open is the definition and evaluation of the quantities s, $\boldsymbol{J}_{S\text{tot}}$, σ. The purely thermodynamic approach (which will be our framework in the present section) goes as follows (de Groot and Mazur 1984, Prigogine 1969, Kreuzer 1981). Equilibrium statistical mechanics provides us (in principle) with an explicit expression of the entropy, valid for any given system, as a function of the state variables, for instance

$$s = s_{\text{eq}}(n_1, \ldots, n_c, T), \tag{5.5}$$

where $n_1, \ldots, n_c$ are the number densities of the various components of the system. "Classical" non-equilibrium thermodynamics starts from a rather bold assumption. It is postulated that, if the system is not "too far away" from equilibrium, the equation of state (5.5) remains valid *locally*. In other words, in any small neighborhood of a point $\boldsymbol{x}$ and at any time t, the functional relationship (5.5) is valid, even though the state variables may vary "slowly" in space and time. Thus, it is asserted that, in the non-equilibrium state,

$$s(\boldsymbol{x}, t) = s_{\text{eq}}(n_1(\boldsymbol{x}, t), \ldots, n_c(\boldsymbol{x}, t), T(\boldsymbol{x}, t)). \tag{5.6}$$

Another way of expressing this *local equilibrium assumption* is to say that the non-equilibrium entropy changes in space and time only through the variation of the state variables. Hence

$$\partial_t s = \sum_\alpha \left(\frac{\partial s_{\text{eq}}}{\partial n_\alpha} \right) \partial_t n_\alpha + \left(\frac{\partial s_{\text{eq}}}{\partial T} \right) \partial_t T. \tag{5.7}$$

Combining this equation with the hydrodynamic balance equations (for $\partial_t n_\alpha$ and $\partial_t T$) results in an expression which, compared to (5.4), yields explicit forms for the entropy flux and for the entropy source.

Clearly, the justification of the hypotheses made here requires a detailed analysis on the basis of kinetic theory. This will be done in chapter 6.

The application of these ideas to our model of a quiescent plasma introduces a few peculiarities. It was already stressed in section 3.2 that the plasma behaves thermodynamically as a classical ideal system. To translate more precisely the idea embodied in the two-fluid picture, it should be said that the equilibrium properties of the plasma are those of a mixture of two independent ideal gases having *different temperatures*. For such a system, classical equilibrium statistical mechanics yields the following expression of the *entropy per particle of species* α [see e.g. ch. 5, eq. (5.2.27) of Balescu (1975)]:

$$s_\alpha = \frac{5}{2} - \ln\left(\frac{h^3}{(2\pi m_\alpha)^{3/2}} \frac{n_\alpha}{T_\alpha^{3/2}} \right), \tag{5.8}$$

where h is the Planck constant. We recall our conventional units where the Boltzmann constant k_B is set equal to 1. The content of eq. (5.8) appears more clearly if we multiply it by $n_\alpha T_\alpha$ and write it in the form

$$T_\alpha n_\alpha s_\alpha = \tfrac{3}{2} n_\alpha T_\alpha + n_\alpha T_\alpha - n_\alpha T_\alpha \ln\left(\frac{h^3}{(2\pi m_\alpha)^{3/2}} \frac{n_\alpha}{T_\alpha^{3/2}} \right). \tag{5.9}$$

We now define the *chemical potential* μ_α of species α as

$$\mu_\alpha = T_\alpha \ln\left(\frac{h^3}{(2\pi m_\alpha)^{3/2}} \frac{n_\alpha}{T_\alpha^{3/2}} \right). \tag{5.10}$$

Recalling eqs. (2.9)–(2.11), we rewrite (5.9) in the form (Misguich and Balescu 1984)

$$T_\alpha n_\alpha s_\alpha = n_\alpha \varepsilon_\alpha + P_\alpha - n_\alpha \mu_\alpha. \tag{5.11}$$

This is typically a *quasi-thermodynamical relation*, in the following sense. For a one-component system, this equation is a well-known thermodynamical formula [App. II, eq. (4) of de Groot and Mazur (1984)]. The peculiarity of (5.11) is the appearance of two distinct temperatures, $T_e \neq T_i$: this point was already underscored in section 3.2. The plasma is not in a true (local) thermodynamical equilibrium state. Rather, because of the smallness of the mass ratio, it lives for a very long time in a quasi-equilibrium state in which the electrons and the ions are separately in equilibrium, at different temperatures. The relaxation towards the true equilibrium (hence the equalization of the temperatures) is a very slow process.

As a result, *separate thermodynamical relations* can be written for each species. The *global entropy density ns* is then defined as

$$ns = \sum_\alpha n_\alpha s_\alpha = \sum_\alpha \frac{1}{T_\alpha} (n_\alpha \varepsilon_\alpha + P_\alpha - n_\alpha \mu_\alpha), \tag{5.12}$$

where $n \equiv \Sigma n_\alpha$ is the total number density. This formula plays the role of the *equation of state* (5.5) in the case of a plasma.

We now pursue the idea of a local equilibrium, as described above, and evaluate the rate of change of the entropy density of species α, by using eq. (5.8),

$$\begin{aligned}\partial_t n_\alpha s_\alpha &= s_\alpha\, \partial_t n_\alpha + n_\alpha\, \partial_t s_\alpha \\ &= s_\alpha\, \partial_t n_\alpha + n_\alpha\left(\tfrac{3}{2}\partial_t \ln T_\alpha - \partial_t \ln n_\alpha\right).\end{aligned}$$

The crucial step, to be performed now, is the combination of this equation with the plasmadynamical balance equations (4.5) and (4.11):

$$\begin{aligned}\partial_t n_\alpha s_\alpha = &-(s_\alpha - 1)\,\nabla\cdot(n_\alpha \boldsymbol{u}^\alpha) \\ &- \frac{1}{T_\alpha}\left(\tfrac{3}{2} n_\alpha \boldsymbol{u}^\alpha\cdot\nabla T_\alpha + n_\alpha T_\alpha\, \nabla\cdot\boldsymbol{u}^\alpha + \boldsymbol{\pi}^\alpha : \nabla \boldsymbol{u}^\alpha + \nabla\cdot\boldsymbol{q}^\alpha - Q^\alpha\right).\end{aligned}$$

This equation [which was also obtained by Braginskii (1965)] is readily transformed into

$$\partial_t n_\alpha s_\alpha = -\nabla\cdot\left(n_\alpha s_\alpha \boldsymbol{u}^\alpha + T_\alpha^{-1}\boldsymbol{q}^\alpha\right) + \sigma^\alpha, \tag{5.13}$$

where

$$\sigma^\alpha = -\frac{1}{T_\alpha^2}\boldsymbol{q}^\alpha\cdot\nabla T_\alpha - \frac{1}{T_\alpha}\boldsymbol{\pi}^\alpha : \nabla\boldsymbol{u}^\alpha + \frac{Q_\alpha}{T_\alpha}. \tag{5.14}$$

Equation (5.13) has the standard form of a balance equation. The *total entropy flux* of species α is therefore identified as

$$\boldsymbol{J}^\alpha_{S\,\mathrm{tot}} = n_\alpha s_\alpha \boldsymbol{u}^\alpha + \frac{1}{T_\alpha}\boldsymbol{q}^\alpha. \tag{5.15}$$

As usual, the total entropy flux contains a convective term, $n_\alpha s_\alpha \boldsymbol{u}^\alpha$, and a conductive entropy flux, equal to the heat flux divided by the temperature of species α.

The source term σ^α is defined by eq. (5.14). It will be called the rate of entropy production of species α per unit volume, or briefly, the *entropy production of species* α.

We now perform a slight transformation. In the electron entropy production appears the electronic heat exchange term Q^{e}, which is related to the corresponding ionic quantity Q^{i} by eq. (4.15). We may therefore write

$$T_{\mathrm{e}}^{-1}Q^{\mathrm{e}} = T_{\mathrm{e}}^{-1}\left(-Q^{\mathrm{i}} - \boldsymbol{u}^{\mathrm{ei}}\cdot\boldsymbol{R}^{\mathrm{ei}}\right) = \frac{1}{en_{\mathrm{e}}T_{\mathrm{e}}}\boldsymbol{j}\cdot\boldsymbol{R}^{\mathrm{ei}} - \frac{Q^{\mathrm{i}}}{T_{\mathrm{e}}}. \tag{5.16}$$

It turns out that the ion heat exchange Q^i is very small [proportional to the mass ratio μ, see eq. (4.6.38)]; it will be most often neglected in this book. The two entropy productions can therefore be rewritten in the less symmetric, but more explicit form

$$\sigma^e = \frac{1}{en_e T_e} \boldsymbol{j} \cdot \boldsymbol{R}^{ei} - \frac{1}{T_e^2} \boldsymbol{q}^e \cdot \nabla T_e - \frac{1}{T_e} \boldsymbol{\pi}^e : \nabla \boldsymbol{u}^e$$

$$\sigma^i = -\frac{1}{T_i^2} \boldsymbol{q}^i \cdot \nabla T_i - \frac{1}{T_i} \boldsymbol{\pi}^i : \nabla \boldsymbol{u}^i. \tag{5.17}$$

The two balance equations (5.13) can also be put together into a global entropy balance,

$$\partial_t ns = -\nabla \cdot \sum_\alpha \left(n_\alpha s_\alpha \boldsymbol{u}^\alpha + T_\alpha^{-1} \boldsymbol{q}^\alpha \right) + \sigma. \tag{5.18}$$

The same remarks can be made here as at the end of section 3.2. The entropies per species, as defined here, are not intimately mixed as in a mixture of neutral gases. For instance, each species' entropy is convected with its own average velocity $n_\alpha s_\alpha \boldsymbol{u}^\alpha$, whereas in the traditional formulation, the convective term is written as $ns\boldsymbol{u}$ (de Groot and Mazur 1984). Of course, a link can be made between the two formulations and explicit transformation formulae can be derived, relating the total entropy flux of (5.18) to the de Groot–Mazur definition of the entropy flux and of the heat flux (Misguich and Balescu 1984). The traditional formulation is not desirable, however, for describing a two-temperature electron–ion plasma, because it hides the fundamental quasi-independence of the "two fluids".

An additional, very important property enhances the motivation of this viewpoint. The second law of thermodynamics requires the inequality

$$\sigma \equiv \sum_\alpha \sigma^\alpha \geqslant 0. \tag{5.19}$$

It will be proved, however, from kinetic theory, that the following stronger property holds:

$$\sigma^\alpha \geqslant 0, \quad \alpha = e, i. \tag{5.20}$$

Thus, *the entropy production for each species is separately positive.*

The treatment given here was purely thermodynamic, i.e. macroscopic. We shall come back to this matter in great detail in Chapters 6 and 17. The

treatment there will be based on kinetic theory. It will allow us to analyze the validity of the assumptions made here and to study the properties of the entropy production in relation to transport theory.

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4

The Hermitian moment representation

4.1. Characteristic time scales. The quasi-neutrality approximation

In order to construct the transport theory, i.e. in order to obtain closed hydrodynamical and electrodynamical equations, we need to solve the kinetic equation (3.3.1). Because of its complexity, an exact solution is practically impossible; therefore, we need approximation methods, valid in the various conceivable regimes. A clue as to the choice of approximation methods is provided by an analysis of the various characteristic times involved in the kinetic equation (see also the discussion in section 2.6).

By a trivial dimensional argument, each term on the right-hand side of (3.3.1) must have a dimension (f^α/T), where T is some characteristic time. But the latter is different for each term.

The *free flow term* introduces a characteristic time τ_H^α which may be called the *hydrodynamical time*. It is related to the spatial variation of the local distribution function. It can be estimated as

$$\frac{1}{\tau_H^\alpha} \sim \frac{V_{T\alpha}}{L_H}, \tag{1.1}$$

where $V_{T\alpha}$ is the thermal velocity of species α, defined in (2.6.13), and L_H is the *length of the gradient* of the density, the velocity or the temperature, whichever is shorter:

$$\frac{1}{L_H} = \text{Max} \frac{|\nabla A(x)|}{|A(x)|}. \tag{1.2}$$

Without going into more precise estimates at this stage (see section 5.1), we note that, in most situations of interest, the gradients are due to some experimental preparation obtained by a macroscopic device. In this case, the length of the gradients is very large compared to molecular dimensions. It

follows that τ_H^α can be considered as a very long time, and therefore the characteristic variations related to the free-flow term are slow *.

The *Vlasov term* of the kinetic equation poses some complicated problems. In the first place, there may be external high-frequency fields present (such as those produced by a laser, or by an RF source used for heating a tokamak, etc.). Such fields will induce fast (but also slow!) motions of the electrons. But even in the absence of external fields, there is a possibility of fast internal motions. Indeed, any fluctuation producing a local charge separation leads to a force on the electrons. The latter start oscillating with the well-known plasma frequency ω_{Pe}, which is, in general, a high frequency. The problem of the coupling of the fast (electrodynamic) motions with the slow (hydrodynamic) motions is a quite interesting, but rather difficult problem. It will not be studied in this first part of our work. We only consider here *slowly varying* electromagnetic fields: the influence of high-frequency phenomena is left for Part III of our work. [A short, but clear discussion of this point is given by Freidberg (1982)].

More specifically, we note that the hydrodynamic length L_H and the hydrodynamic time τ_H were defined in (1.2) through the characteristic variation of the hydrodynamical quantities ρ, $\boldsymbol{u}$, and T_α. We now assume that L_H, τ_H *are also the characteristic length and time scales of the electric field* $\boldsymbol{E}$ *and of the magnetic field* $\boldsymbol{B}$.

This approximation may seem incorrect because of the existence, even in the absence of external high-frequency fields, of the high-frequency waves discussed above. However, in a linear theory, these effects do not influence the slow plasma motions in which we are interested here. This can be understood from the following schematic argument.

All the hydrodynamic quantities as well as the electromagnetic fields may be split into a slowly varying part A_S and a rapidly varying part part A_F: $A = A_S + A_F$. Their equations of evolution can be written in the form

$$\partial_t A_S = \Phi_S(A_S, A_F), \qquad \partial_t A_F = \Phi_F(A_S, A_F).$$

Here A_S, A_F should be considered as a kind of vectors, representing the set of all relevant quantities; the right-hand sides represent some, generally non-linear, combinations of both slow and fast variables and of their gradients. If these equations are *linearized* as in classical transport theory, it is clear that the slow (fast) variables can only be determined by the slow (fast) variables in

* A notable exception can occur in laser-created plasmas. The density gradient near the critical surface may become very steep, with L_H of the order of microns. This case must be discussed separately.

the right-hand side. The evolution of the slow variables can then be studied independently of the fast ones, as will indeed be done here. If, however, non-linear effects become important (e.g. when the amplitude of the fast electromagnetic fields is high, as for instance in the presence of an intense laser beam or of an intense RF heating beam in a tokamak), two high-frequency components may couple and produce a low-frequency beat-wave, which influences the slow motions. This type of non-linear coupling will be studied in Part III of this work.

If the main assumption is accepted, the well-known *pre-Maxwell approximation* follows very easily (see Priest 1982). Indeed, the Maxwell equation (3.4.24) leads to the following estimate of the ratio of the electric and magnetic fields:

$$\frac{|E|}{L_H} \approx \frac{1}{c}\frac{|B|}{\tau_H}. \tag{1.3}$$

On the other hand, (3.4.26) yields

$$\frac{|B|}{L_H} \approx \frac{1}{c}\frac{|E|}{\tau_H} + \frac{4\pi}{c}|j|. \tag{1.4}$$

Combining these two equations, we find that the displacement current term is of order $L_H^2/\tau_H^2 c^2$ compared to the left-hand side. As L_H/τ_H is of the order of the hydrodynamic velocity (by our "main assumption"), we may conclude

$$\left|\frac{1}{c}\partial_t \boldsymbol{E}\right| \approx \frac{|u|^2}{c^2}|\nabla \wedge \boldsymbol{B}|. \tag{1.5}$$

Hence, whenever the motion of the plasma is non-relativistic ($u^2/c^2 \ll 1$), the displacement current can be neglected and the (slowly varying) fields obey the *pre-Maxwell equations*

$$\nabla \cdot \boldsymbol{E} = 4\pi\sigma, \tag{1.6}$$

$$\nabla \wedge \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B}, \tag{1.7}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.8}$$

$$\nabla \wedge \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}. \tag{1.9}$$

The tricky point in the pre-Maxwell approximation is the status of the *charge balance equation* (3.4.29). This equation was derived from the kinetic equation; on the other hand, it is *also* a consequence of the *complete* set of Maxwell equations. In the pre-Maxwell approximation, the truncated equation (3.4.26) requires the current density to obey the constraint

$$\nabla \cdot \boldsymbol{j} \approx 0. \tag{1.10}$$

But the complete charge balance equation is still valid (as a consequence of the kinetic equation); in order to ensure the mutual compatibility of these conditions, we necessarily must have

$$\partial_t \sigma(\boldsymbol{x}, t) \approx 0. \tag{1.11}$$

Equation (1.11) should not, however, be considered as an exact equation; it only means that the charge density is a slowly varying function. As for the *size* of this quantity, it may be estimated from the following argument. It is well known that a plasma (or any form of matter) cannot support large local charge separations. It is therefore almost always true that, in all points $\boldsymbol{x}$ and at all times t,

$$|n_e(\boldsymbol{x}, t) - n_i(\boldsymbol{x}, t)| \ll \tfrac{1}{2}\left[n_e(\boldsymbol{x}, t) + n_i(\boldsymbol{x}, t)\right]. \tag{1.12}$$

This will be called the *local quasi-neutrality condition*. It implies (see table 3.2.2)

$$\sigma(\boldsymbol{x}, t) \ll \frac{e}{m_i}\rho(\boldsymbol{x}, t) = en_i(\boldsymbol{x}, t),$$

$$\sigma(\boldsymbol{x}, t) \ll \frac{Ze}{m_i}\rho(\dot{\boldsymbol{x}}, t) = en_e(\boldsymbol{x}, t). \tag{1.13}$$

As a result, a glance at eqs. (3.4.21), (3.4.22) and (3.4.27) shows that *the charge density σ can be neglected in all the hydrodynamical equations*, except for the term $\sigma \boldsymbol{E}$ in (3.4.21). This term is, however, negligible for a different reason. A dimensional argument similar to eqs. (1.3)–(1.5) leads to the estimate

$$\frac{\sigma E}{c^{-1}|\boldsymbol{j} \wedge \boldsymbol{B}|} \approx \frac{(\nabla \cdot \boldsymbol{E})E}{|\nabla \wedge \boldsymbol{B}|B} \approx \frac{L_H^{-1}E^2}{L_H^{-1}B^2} \approx \frac{u^2}{c^2}. \tag{1.14}$$

The last estimate follows from (1.3). Thus, in the present, non-relativistic

approximation, *the term σE must be suppressed from the equation of motion*, for consistency with the pre-Maxwell approximation.

Next, we assume that

$$|\sigma(x, t)\, u(x, t)| \ll |j(x, t)|. \tag{1.15}$$

This means that the convection current is much smaller than the conduction current. This assumption is reasonable in view of eq. (1.13), but is not a strict consequence of the latter. When eq. (1.15) is satisfied, one sees that *all the terms involving the charge density σ can also be neglected in the generalized Ohm law* (3.4.30). Moreover, the exact relations in table 3.2.2, column A, between one-fluid and two-fluid quantities are replaced in the present conditions by the much simpler relations in column C of the same table.

The only place where σ cannot be neglected is the *Poisson equation* (1.6). This equation, however, acquires a different status. Indeed, the generalized Ohm law (derived from kinetic theory) can be used for completely determining the electric field. Hence the Poisson equation becomes superfluous: it only serves for defining the charge density – a quantity which does not influence the motion of the plasma in the present approximation. For a further discussion of these questions, see chapter 7.

We now come to the last term of (3.3.1). It is known from kinetic theory that the effect of the collisions is a uniform relaxation towards thermal equilibrium. This process takes place in a characteristic time τ_α, the *relaxation time*. One also often uses the *collision frequency* $\nu_\alpha = \tau_\alpha^{-1}$ as a measure of the efficiency of the collision term. The relaxation time is an internal, molecular parameter, whose value depends on the state of the plasma (density and temperature) *.

In Part I of our work we study *collision-dominated plasmas*. By this phrase, we mean a plasma for which *the relaxation time is much shorter than the hydrodynamical time*. As will be seen, this situation is the one which leads to the most "classical" type of transport phenomena, i.e. to hydrodynamical equations which are very close (though not identical) to those of ordinary fluids.

4.2. *The local plasma equilibrium state*

Under the conditions where the plasma is dominated by collisions, we may mentally subdivide the evolution process into two phases. Starting from an

* More precisely, there is an infinite number of relaxation times associated with $\mathscr{K}^\alpha$: the set of eigenvalues of the linearized collision operation. *The* relaxation time is, by definition, the longest of these times.

arbitrary initial state, the collisions would tend – if they were alone – to bring the system very quickly to a stationary state: the *thermal equilibrium*. But the slow processes – free flow and electromagnetic processes – prevent the plasma from reaching this state. The result is that, after a short time, of the order of the relaxation time τ_α, the plasma reaches a state very close to the equilibrium. From here on, the distribution functions evolve on the slow time scale.

We now make these statements more precise. If the slow processes could be neglected, the kinetic equation would reduce to

$$\partial_t f^\alpha = \mathscr{K}^\alpha \equiv \mathscr{K}^{\alpha\alpha} + \mathscr{K}^{\alpha\alpha'}, \tag{2.1}$$

where α' is the complement of α, i.e. if $\alpha = \mathrm{e}$, $\alpha' = \mathrm{i}$ and conversely.

These equations describe a monotonic approach towards the equilibrium state, which is defined as the stationary solution of (2.1),

$$\mathscr{K}^\alpha = 0. \tag{2.2}$$

The solution of this equation is well known: the distribution functions are *Maxwellian*,

$$f^{\alpha\,\mathrm{eq}} = n_\alpha \left(\frac{m_\alpha}{2\pi T}\right)^{3/2} \exp\left(-\frac{m_\alpha |\boldsymbol{v}-\boldsymbol{u}|^2}{2T}\right). \tag{2.3}$$

The Maxwellians may be centered around an average non-zero velocity $\boldsymbol{u}$. But, to satisfy (2.2), it is necessary that both the velocity $\boldsymbol{u}$ and the temperature T be the same for the electrons and for the ions. If we substitute this function into (**3**.3.8), we note that

$$G_{mn}(\boldsymbol{g})\left(m_\alpha^{-1}\,\partial_{1n} - m_\beta^{-1}\,\partial_{2n}\right) f^{\alpha\,\mathrm{eq}}(\boldsymbol{v}_1)\, f^{\beta\,\mathrm{eq}}(\boldsymbol{v}_2)$$

$$= -\frac{1}{T} G_{mn}(\boldsymbol{g})\,(v_{1n} - v_{2n}) f^{\alpha\,\mathrm{eq}}(\boldsymbol{v}_1)\, f^{\beta\,\mathrm{eq}}(\boldsymbol{v}_2) = 0 \tag{2.4}$$

the conclusion following from identity (**2**.7.8). We have thus found the stationary solution of (2.3): it can moreover be shown that it is the only such solution (see, e.g. Balescu 1975). But this does not end our discussion.

We recall that the distribution functions considered here are *local* distribution functions, i.e.

$$f^{\alpha\,\mathrm{eq}} = f^{\alpha\,\mathrm{eq}}(\boldsymbol{v};\ \boldsymbol{x},\ t).$$

With this fact in mind, we note that the argument of eq. (2.4) is not altered if the parameters of the Maxwellian, i.e. n_α, $\boldsymbol{u}$, T are functions of $\boldsymbol{x}$ and t. Such a state is called in kinetic theory a *local equilibrium state*. It must be clearly realized that this is *not* a state of thermodynamic equilibrium, because the latter must be homogeneous and stationary. The local equilibrium state satisfies eq. (2.2), but not $\partial_t f^\alpha = 0$. There is, however, no contradiction involved, if we recall that (2.1) is *not* the complete kinetic equation. The space- and time-dependence of the parameters n_α, $\boldsymbol{u}$, T will be governed by the slow processes of evolution.

The previous discussion is standard in the kinetic theory of gases. In the case of plasmas, however, it misses an important point: the fact that the various types of collisions are not equally efficient. Whereas collisions between particles of equal mass are quite effective in redistributing momentum and energy among the partners, collisions between unlike particles of largely different mass behave very poorly from this point of view. As a result, the relaxation towards the equilibrium, described by (2.1), proceeds in two stages. First, the like-particle collisions bring each component to its own state of local equilibrium. As this "partial equilibration" proceeds independently for the two species, nothing prevents the two components from having different temperatures and average velocities. Afterwards, the temperatures and average velocities tend to equilibrate, but this process is much slower, especially for the temperatures: the time scale involved is of the same order as the slow time scales related to the free-flow and the electromagnetic processes. Thus, in a sense, *the unlike-particle collisions must be treated on the same footing as the slow processes*.

To sum up, the separation of fast and slow processes in the present case must be done as follows. The electron–electron and the ion–ion collisions bring the plasma in a short time to a state of *local plasma equilibrium*, satisfying the equations

$$\mathscr{K}^{\mathrm{ee}} = 0, \qquad \mathscr{K}^{\mathrm{ii}} = 0. \tag{2.5}$$

The local distribution functions satisfying these equations are

$$f^{\alpha 0}(\boldsymbol{v};\, \boldsymbol{x},\, t) = n_\alpha(\boldsymbol{x},\, t)\left(\frac{m_\alpha}{2\pi T_\alpha(\boldsymbol{x},\, t)}\right)^{3/2} \exp\left(-\frac{m_\alpha |\, \boldsymbol{v} - \boldsymbol{u}^\alpha(\boldsymbol{x},\, t)\,|^2}{2T_\alpha(\boldsymbol{x},\, t)}\right). \tag{2.6}$$

In this state, the density, the average velocity and the temperature of the electrons and of the ions can be different. We must recall, however, the local

electroneutrality constraint (1.12) which implies the following relation between ion and electron densities:

$$n_e(x, t) \simeq Zn_i(x, t). \tag{2.7}$$

It is important to note that the parameters of the local plasma equilibrium state are none other than the *two-fluid variables* on which macroscopic plasmadynamics is based.

The subsequent stages of the evolution are governed by the slow processes (including the unlike-particle collisions). The role of the fast collisions $\mathscr{K}^{\alpha\alpha}$ is to keep the system constantly in the neighbourhood of the plasma local equilibrium.

Just as in the previous section, the discussion here was largely qualitative. The precise quantitative criteria for the validity of the concepts discussed here will emerge a posteriori.

4.3. The Hermitian moment expansion

As follows from the discussion in section 4.2, the state of the plasma, after a short transition time, remains close to the local plasma equilibrium. For this reason, the local plasma equilibrium will be called a *reference state*. The distribution functions can then conveniently be written in the form

$$f^\alpha(v; x, t) = f^{\alpha 0}(v; x, t)\,[1 + \hat{\chi}^\alpha(v; x, t)]. \tag{3.1}$$

The functions $\hat{\chi}^\alpha(v; x, t)$ measure the deviation of the local distribution function f^α from the reference state. As the latter is a known function, it is clear that the whole information about the real state of the plasma is contained in $\hat{\chi}^\alpha$.

We now introduce a condition which is very useful in providing an unambiguous interpretation of the results. We have seen that the local plasma equilibrium distribution function (2.6) depends on five parameters: n_α, $\boldsymbol{u}^\alpha$, T_α (for each species). If the plasma is in a state of local equilibrium, these parameters coincide precisely with the two-fluid variables defined in table 3.2.1.

But in an arbitrary state (3.1), this coincidence no longer necessarily holds. It is, however, possible to construct representation (3.1) in such a way that the parameters n_α, u_r^α, T_α entering the definition of the reference state do coincide

with the *exact* values of the density, velocity and temperature of each species. This implies

$$\int \mathrm{d}\boldsymbol{v} f^{\alpha}(\boldsymbol{v};\, \boldsymbol{x},\, t) = \int \mathrm{d}\boldsymbol{v} f^{\alpha 0}(\boldsymbol{v};\, \boldsymbol{x},\, t) = n_{\alpha}(\boldsymbol{x},\, t),$$

$$\int \mathrm{d}\boldsymbol{v}\, v_r f^{\alpha}(\boldsymbol{v};\, \boldsymbol{x},\, t) = \int \mathrm{d}\boldsymbol{v}\, v_r f^{\alpha 0}(\boldsymbol{v};\, \boldsymbol{x},\, t) = n_{\alpha}(\boldsymbol{x},\, t) u_r^{\alpha}(\boldsymbol{x},\, t),$$

$$\tfrac{1}{3} m_{\alpha} \int \mathrm{d}\boldsymbol{v}\, |\boldsymbol{v} - \boldsymbol{u}^{\alpha}|^2 f^{\alpha}(\boldsymbol{v};\, \boldsymbol{x},\, t) = \tfrac{1}{3} m_{\alpha} \int \mathrm{d}\boldsymbol{v}\, |\boldsymbol{v} - \boldsymbol{u}^{\alpha}|^2 f^{\alpha 0}(\boldsymbol{v};\, \boldsymbol{x},\, t)$$

$$= n_{\alpha}(\boldsymbol{x},\, t)\, T_{\alpha}(\boldsymbol{x},\, t). \tag{3.2}$$

These conditions can be reformulated as *constraints* on the deviations:

$$\int \mathrm{d}\boldsymbol{v} f^{\alpha 0} \hat{\chi}^{\alpha} = 0, \qquad \int \mathrm{d}\boldsymbol{v} f^{\alpha 0}\, v_r\, \hat{\chi}^{\alpha} = 0, \qquad \int \mathrm{d}\boldsymbol{v} f^{\alpha 0}\, v^2 \hat{\chi}^{\alpha} = 0. \tag{3.3}$$

The following picture emerges now. During its evolution, the plasma goes through successive states which remain close to the reference state. The latter contains all the information about the plasmadynamical variables n_{α}, $\boldsymbol{u}^{\alpha}$, T_{α}. The reference state is itself slowly varying in space and time, because of the variation of the latter parameters.

Before continuing, it is convenient to note that the distribution functions $f^{\alpha 0}$ depend on the velocity $\boldsymbol{v}$ only through the dimensionless variable

$$\boldsymbol{c} = \left(\frac{m_{\alpha}}{T_{\alpha}(\boldsymbol{x},\, t)} \right)^{1/2} [\boldsymbol{v} - \boldsymbol{u}^{\alpha}(\boldsymbol{x},\, t)]. \tag{3.4}$$

As most of the important quantities of table 3.2.1. are averages of powers of the relative velocities $\boldsymbol{v} - \boldsymbol{u}^{\alpha}$, it is clear that $\boldsymbol{c}$ is a quite convenient variable. We shall therefore express all functions in terms of it. Clearly, some care must be taken by noting that the passage from $\boldsymbol{v}$ to $\boldsymbol{c}$ is different for the electrons and for the ions, and that the coefficients of the transformation depend on $\boldsymbol{x}$ and t. We now write the *local plasma equilibrium distribution* as

$$f^{\alpha 0}(\boldsymbol{v};\, \boldsymbol{x},\, t) = n_{\alpha}(\boldsymbol{x},\, t) \left(\frac{m_{\alpha}}{T_{\alpha}(\boldsymbol{x},\, t)} \right)^{3/2} \phi^{\alpha 0}(\boldsymbol{c};\, \boldsymbol{x},\, t), \tag{3.5}$$

where

$$\phi^{\alpha 0}(\boldsymbol{c};\, \boldsymbol{x},\, t) \equiv \phi^{0}(c) = (2\pi)^{-3/2} \exp(-c^2). \tag{3.6}$$

We note the very important fact that *the reference state, when expressed in terms of the variables* $\boldsymbol{c}$*, is a stationary, homogeneous and isotropic state*, whereas considered as a function of the original variable $\boldsymbol{v}$, it has none of these properties. Moreover, $\phi^{\alpha 0}(c)$ is the same function for the electrons and for the ions, it is therefore independent of the superscript α. The dimensionless reference distribution function is normalized as

$$\int \mathrm{d}\boldsymbol{c}\ \phi^0(c) = 1. \tag{3.7}$$

We now express the complete distribution functions (3.1) as

$$f^\alpha(\boldsymbol{v};\ \boldsymbol{x},\ t) = n_\alpha(\boldsymbol{x},\ t)\left(\frac{m_\alpha}{T_\alpha(\boldsymbol{x},\ t)}\right)^{3/2} \phi^\alpha(\boldsymbol{c};\ \boldsymbol{x},\ t)$$

$$= n_\alpha(\boldsymbol{x},\ t)\left(\frac{m_\alpha}{T_\alpha(\boldsymbol{x},\ t)}\right)^{3/2} \phi^0(c)\ [1 + \chi^\alpha(\boldsymbol{c};\ \boldsymbol{x},\ t)]. \tag{3.8}$$

The constraints, expressed in the new variables, are

$$\int \mathrm{d}\boldsymbol{c}\ \phi^0 \chi^\alpha = 0, \qquad \int \mathrm{d}\boldsymbol{c}\ \phi^0\ c_r\ \chi^\alpha = 0, \qquad \int \mathrm{d}\boldsymbol{c}\ \phi^0\ c^2\ \chi^\alpha = 0. \tag{3.9}$$

We now start the investigation of our main problem, which consists of finding an approximate solution for the unknown functions χ^α. Several methods have been developed for this purpose. The most ancient and celebrated one is the Chapman–Enskog method, developed independently by these authors already in 1916 (Chapman and Cowling 1952). Grad's (1949) moment method has also been widely used. Finally, Résibois's (1970) more recent projection operator method is quite elegant and useful in the linear domain. All these methods, in various versions, have been applied to the problem of transport in plasmas. We shall give in section 5.7 a comparative discussion of the results obtained by various methods. Here, we start directly with a method which has – in our opinion – the advantage of clarity and simplicity. It is close in spirit to Grad's method but has features in common with the two others as well.

The general idea is quite straightforward. Adopting a technique widely used in quantum mechanics, we start by expanding the unknown functions χ^α in a series of orthogonal polynomials. The determination of the coefficients of this series is equivalent to the determination of χ^α. The kinetic equation provides an infinite set of equations for these coefficients. If we are clever (lucky?) the

series will rapidly converge. A truncated set of equations will then be sufficient for the determination of the transport properties with the required precision.

The first step consists of choosing an adequate set of *orthogonal polynomials*. We want an expansion of $\chi^\alpha(\boldsymbol{c};\ \boldsymbol{x},\ t)$, considered as a function of $\boldsymbol{c}$, i.e. o the three scalar variables c_x, c_y, c_z. As the local equilibrium distribution (3.6) obviously plays a crucial role in the theory, a very natural basis is provided by the *Hermite polynomials*, which are orthogonal with respect to a Gaussian weight function. As we are considering here functions of *three* variables, we need to use the so-called *tensorial Hermite polynomials*, a generalization of the well-known one-dimensional Hermite polynomials. But even at this level, we still have a choice. The problems arising in this context are discussed in detail in the General Appendix G1. It is shown there that the use of the straightforward three-dimensional generalization of the Hermite polynomials ("*reducible tensorial Hermite polynomials*") (as was done by Grad 1949) leads to a rather untransparent expansion.

It turns out that a representation in which the various types of anisotropy of the distribution function are clearly exhibited, is the most convenient guide for reasonable truncation approximations. Such a representation would be of the form

$$\chi^\alpha(\boldsymbol{c};\ \boldsymbol{x},\ t) = A^\alpha(c;\ \boldsymbol{x},\ t) + c_r B_r^\alpha(c;\ \boldsymbol{x},\ t) + \left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right) C_{rs}^\alpha(c;\ \boldsymbol{x},\ t) + \cdots, \tag{3.10}$$

where A^α is a scalar function, B_r^α a vector function, C_{rs}^α a symmetric traceless tensor function, etc. All of these depend only on the absolute value of the variable $\boldsymbol{c}$, as well as on the variables $\boldsymbol{x}$ and t. These functions are sometimes called "*anisotropies*" of successive orders. Representation (3.10) is taken as a basic Ansatz in the Chapman–Enskog method (Chapman and Cowling 1952, Braginskii 1965, Ferziger and Kaper 1972). In that method, the functions A^α, B_r^α, C_{rs}^α are then further expanded in series of Laguerre–Sonine polynomials of the single variable c, and each of them is truncated at some appropriate level (see General Appendix G1).

The reason for the usefulness of such a representation is clear. Our main use of the functions χ^α will be in the calculation of the relevant fluxes appearing in the hydrodynamical balance equations, such as the heat flux q_r^α and the pressure tensor π_{rs}^α. In a linear theory, the former will be entirely determined by the *vector* part of χ^α, i.e. by B_r^α, whereas the pressure tensor involves only the tensor part of χ^α. Even in a non-linear theory, the explicit exhibition of the anisotropies helps considerably in the calculations.

It is shown in the General Appendix G1 that the most convenient expansion, which exhibits explicitly the anisotropies, is obtained by using the

irreducible tensorial Hermite polynomials, $H^{(m)}_{r_1 \dots r_q}(\boldsymbol{c})$. The deviation of the distribution function is then represented as in eq. (**G1**.4.3),

$$\chi^{\alpha}(\boldsymbol{c};\, \boldsymbol{x},\, t) = \sum_{n=0}^{\infty} h^{\alpha(2n)}(\boldsymbol{x},\, t)\; H^{(2n)}(\boldsymbol{c})$$

$$+ \sum_{n=0}^{\infty} h_r^{\alpha(2n+1)}(\boldsymbol{x},\, t)\; H_r^{(2n+1)}(\boldsymbol{c})$$

$$+ \sum_{n=1}^{\infty} h_{rs}^{\alpha(2n)}(\boldsymbol{x},\, t)\; H_{rs}^{(2n)}(\boldsymbol{c}) + \cdots . \tag{3.11}$$

The function $\chi^{\alpha}(\boldsymbol{c})$ of the continuous vector variable $\boldsymbol{c}$ is thus represented by a denumerable infinite set of coefficients $h^{\alpha(m)}_{\dots}$ called the *(irreducible) Hermitian moments* of the distribution function; these, in turn, are classified as *scalar Hermitian moments* $h^{\alpha(2n)}$, *vector Hermitian moments* $h_r^{\alpha(2n+1)}$, *traceless tensor Hermitian moments* $h_{rs}^{\alpha(2n)}$, etc. The moments are related to the original function by eq. (**G1**.4.12) as

$$h^{\alpha(m)}_{r_1 \dots r_q}(\boldsymbol{x},\, t) = \int \mathrm{d}\boldsymbol{c}\; \phi^0(c)\; H^{(m)}_{r_1 \dots r_q}(\boldsymbol{c})\; \chi^{\alpha}(\boldsymbol{c};\, \boldsymbol{x},\, t). \tag{3.12}$$

Alternatively, the moment $h^{\alpha(m)}_{r_1 \dots r_q}$ (for $m \neq 0$) can also be interpreted as the *average value of the Hermite polynomial* $H^{(m)}_{r_1 \dots r_q}(\boldsymbol{c})$, calculated with the *full (dimensionless) distribution function* $\phi^{\alpha}(\boldsymbol{c};\, \boldsymbol{x},\, t)$, eq. (3.8),

$$h^{\alpha(m)}_{r_1 \dots r_q}(\boldsymbol{x},\, t) = \int \mathrm{d}\boldsymbol{c}\; \phi^0(c)\; [1 + \chi^{\alpha}(\boldsymbol{c};\, \boldsymbol{x},\, t)]\, H^{(m)}_{r_1 \dots r_q}(\boldsymbol{c}), \quad m \neq 0. \tag{3.13}$$

Indeed, the orthogonality of any polynomial $H^{(m)}_{\dots}$ to $H^{(0)} \equiv 1$ implies

$$(2\pi)^{-3/2} \int \mathrm{d}\boldsymbol{c}\; \exp\left(-\tfrac{1}{2}c^2\right) H^{(m)}_{r_1 \dots r_q}(\boldsymbol{c}) = \delta_{m,0}. \tag{3.14}$$

Hence, *in the local equilibrium state, the average value of all Hermite polynomials of non-zero degree is identically zero.*

The irreducible tensorial Hermite polynomials are defined in the General Appendix G1, eqs. (**G1**.4.5)–(**G1**.4.7). The first members of each set are

displayed in table **G1**.4.1. It is seen that the tensorial polynomials are of the form

$$H_r^{(2n+1)}(\boldsymbol{c}) = c_r f^{(2n+1)}(c),$$

$$H_{rs}^{(2n)}(\boldsymbol{c}) = \left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right) g^{(2n)}(c), \tag{3.15}$$

where $f^{(2n+1)}(c)$, $g^{(2n)}(c)$ are functions of the absolute value of the vector $\boldsymbol{c}$. Hence, when written in terms of the irreducible Hermite polynomials, the expansion (3.11) is precisely of the form (3.10).

Next, we must be sure that the constraints (3.9) are satisfied. Because of the orthogonality of the Hermite polynomials, the constraints take a very simple form. They amount to requiring that three Hermitian moments by identically zero for every acceptable deviation χ^α:

$$h^{\alpha(0)} \equiv 0, \qquad h^{\alpha(2)} \equiv 0, \qquad h_r^{\alpha(1)} \equiv 0. \tag{3.16}$$

Finally, we note that the moments $h_{rs}^{\alpha(2)}$ and $h_r^{\alpha(3)}$ have a very important physical meaning. Using table **3**.2.1, we have

$$\pi_{rs}^\alpha = m_\alpha \int \mathrm{d}\boldsymbol{v}\, \left[(v_r - u_r^\alpha)(v_s - u_s^\alpha) - \tfrac{1}{3}|\boldsymbol{v} - \boldsymbol{u}^\alpha|^2 \delta_{rs}\right] f^\alpha$$

$$= m_\alpha \frac{T_\alpha}{m_\alpha} n_\alpha \int \mathrm{d}\boldsymbol{c}\, \left[c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right] \phi^0 [1 + \chi^\alpha]$$

$$= T_\alpha n_\alpha \sqrt{2} \int \mathrm{d}\boldsymbol{c}\, H_{rs}^{(2)}\, \phi^0\, \chi^\alpha.$$

Hence, we find a very simple relation between the *pressure tensor* and the *second-order tensor Hermitian moment*:

$$\pi_{rs}^\alpha = \sqrt{2}\, n_\alpha T_\alpha\, h_{rs}^{\alpha(2)}. \tag{3.17}$$

We easily derive a similar relation between the *heat flux* and the *third-order vector Hermitian moment*:

$$q_r^\alpha = \sqrt{\tfrac{5}{2}}\, m_\alpha \left(\frac{T_\alpha}{m_\alpha}\right)^{3/2} n_\alpha\, h_r^{\alpha(3)}. \tag{3.18}$$

We have now achieved the desired series expansion, exhibiting the tensorial symmetries explicitly. For calculational purposes, these series must be truncated. This truncation process is done in two stages. As said before, in a linear theory we shall only need the vector and traceless second-rank tensor parts. The deeper reason for retaining only these two classes of moments will appear gradually as we go on: see particularly section **5**.2 and chapter 6. In particular, it is shown in section **6**.3 that the scalars and the anisotropies of order three and higher give rigorously no contribution to the entropy production (in the linear regime). We thus neglect all other anisotropies and write

$$\chi^{\alpha}(\boldsymbol{c};\ \boldsymbol{x},\ t) \simeq c_r B_r^{\alpha}(c;\ \boldsymbol{x},\ t) + \left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right) C_{rs}^{\alpha}(c;\ \boldsymbol{x},\ t). \tag{3.19}$$

The series for B_r^{α} and C_{rs}^{α} can then be truncated at various levels. We consider here three successive approximations.

(A) The thirteen moment (13 M) approximation *:

$$c_r B_r^{\alpha}(c;\ \boldsymbol{x},\ t) = h_r^{\alpha(3)}(\boldsymbol{x},\ t)\ H_r^{(3)}(\boldsymbol{c}),$$

$$\left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right) C_{rs}^{\alpha}(c;\ \boldsymbol{x},\ t) = h_{rs}^{\alpha(2)}(\boldsymbol{x},\ t)\ H_{rs}^{(2)}(\boldsymbol{c}). \tag{3.20}$$

(B) The twenty-one moment (21 M) approximation:

$$c_r B_r^{\alpha}(\boldsymbol{c};\ \boldsymbol{x},\ t) = h_r^{\alpha(3)}(\boldsymbol{x},\ t)\ H_r^{(3)}(\boldsymbol{c}) + h_r^{\alpha(5)}(\boldsymbol{x},\ t)\ H_r^{(5)}(\boldsymbol{c}),$$

$$\left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right) C_{rs}^{\alpha}(\boldsymbol{c};\ \boldsymbol{x},\ t) = h_{rs}^{\alpha(2)}(\boldsymbol{x},\ t)\ H_{rs}^{(2)}(\boldsymbol{c}) + h_{rs}^{\alpha(4)}(\boldsymbol{x},\ t)\ H_{rs}^{(4)}(\boldsymbol{c}). \tag{3.21}$$

(C) The twenty-nine moment (29 M) approximation:

$$c_r B_r^{\alpha}(\boldsymbol{c};\ \boldsymbol{x},\ t) = h_r^{\alpha(3)}(\boldsymbol{x},\ t)\ H_r^{(3)}(\boldsymbol{c}) + h_r^{\alpha(5)}(\boldsymbol{x},\ t)\ H_r^{(5)}(\boldsymbol{c}) + h_r^{\alpha(7)}(\boldsymbol{x},\ t)\ H_r^{(7)}(\boldsymbol{c}),$$

$$\left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right) C_{rs}^{\alpha}(\boldsymbol{c};\ \boldsymbol{x},\ t) = h_{rs}^{\alpha(2)}(\boldsymbol{x},\ t)\ H_{rs}^{(2)}(\boldsymbol{c}) + h_{rs}^{\alpha(4)}(\boldsymbol{x},\ t)\ H_{rs}^{(4)}(\boldsymbol{c}) + h_{rs}^{\alpha(6)}(\boldsymbol{x},\ t)\ H_{rs}^{(6)}(\boldsymbol{c}). \tag{3.22}$$

* In this approximation one retains, for each species, the 5 plasmadynamical moments n_α, u_r^α and T_α (contained in $f^{\alpha 0}$), plus 3 components of the heat flux $h_r^{\alpha(3)}$, plus the 5 independent components of the pressure tensor $h_{rs}^{\alpha(2)}$. These sum up to 13 moments.

It will be seen, a posteriori, that in a linear theory, the 13 M approximation is rather poor, whereas the 21 M approximation is excellent. The 29 M approximation adds no significant contribution to the transport coefficients (see section 5.4).

4.4. *Classification of the moments*

Having defined the nature of the Hermitian moment expansion, we now pause a little while, before continuing the systematic development of the transport theory for collision-dominated plasmas. In order to get a clear picture of the macroscopic, as well as of the microscopic description of the plasma, it is useful to discuss globally the properties of the various moments entering the theory. Many of the points appearing in the present section were already discussed previously; nevertheless, it is good to put them all together here.

The various moments necessary for a description of the plasma can be classified in at least two ways, according to the criteria chosen for this operation (see section 3.2). We begin our discussion with the *two-fluid picture*.

In a first group we put the *plasmadynamical moments*, n_α, $\boldsymbol{u}^\alpha$, T_α ($\alpha = \mathrm{e}, \mathrm{i}$). They are defined as *moments of the reference local equilibrium distribution* $f^{\alpha 0}$, *even in a non-equilibrium state*. This follows from the constraints (3.2) and (3.3) imposed on the distribution functions. Conversely, it may be said that the knowledge of the plasmadynamical fields $n_\alpha(\boldsymbol{x}, t)$, $\boldsymbol{u}^\alpha(\boldsymbol{x}, t)$ and $T_\alpha(\boldsymbol{x}, t)$ entirely determines the reference state.

In a second group we put the *nonplasmadynamical moments*. They are defined as the set of *all Hermitian moments of the distribution functions*, $h^{\alpha(m)}_{\cdots}(\boldsymbol{x}, t)$. They have a property which is complementary to the plasmadynamical variables: *their value is identically zero in the reference local equilibrium state*, as follows from (3.14). Thus, the knowledge of all the non-plasmadynamical moments completely determines the deviation χ^α of the real state from the reference state, as follows clearly from (3.11).

We may also use a different criterion for the classification of the moments, which corresponds to the *one-fluid* picture of the plasma. This amounts to a "reshuffling" of the plasmadynamical variables. Instead of the two scalar moments n_e, n_i, one introduces the linear combinations ρ and σ:

$$\begin{pmatrix} n_\mathrm{e} \\ n_\mathrm{i} \end{pmatrix} \Rightarrow \begin{pmatrix} \rho = \sum_\alpha m_\alpha n_\alpha \\ \sigma = \sum_\alpha e_\alpha n_\alpha \end{pmatrix}. \tag{4.1}$$

Similarly, the two vector moments $\boldsymbol{u}^e$, $\boldsymbol{u}^i$ are replaced by the two linear combinations

$$\begin{pmatrix} \boldsymbol{u}^e \\ \boldsymbol{u}^i \end{pmatrix} \Rightarrow \begin{pmatrix} \rho \boldsymbol{u} = \sum_\alpha m_\alpha n_\alpha \boldsymbol{u}^\alpha \\ \boldsymbol{j} = \sum_\alpha e_\alpha n_\alpha \boldsymbol{u}^\alpha \end{pmatrix}. \tag{4.2}$$

In some cases, it turns out that the electric current $\boldsymbol{j}$ is not a useful quantity (a typical instance arises in the neoclassical theory of radial transport in a magnetically confined plasma). One may then use an alternative choice of moments, by considering the mass flux $\rho \boldsymbol{u}$ and, say, the electron flux, Γ^e,

$$\begin{pmatrix} \boldsymbol{u}^i \\ \boldsymbol{u}^e \end{pmatrix} \Rightarrow \begin{pmatrix} \rho \boldsymbol{u} = \sum_\alpha m_\alpha n_\alpha \boldsymbol{u}^\alpha \\ \Gamma^e = n_e \boldsymbol{u}^e \end{pmatrix}. \tag{4.3}$$

In the new, one-fluid picture, we put together the mass density ρ and the mass flux $\rho \boldsymbol{u}$ (which is identical to the momentum density): they are averages of *conserved quantities* (i.e. collisional invariants, see section 2.6). In the same group we put the two temperatures T_α, which are quasi-conserved quantities. This means that the collisional heat exchange between the two species is negligibly small, as will be shown at the end of section 4.6. The set of quantities ρ, $\rho \boldsymbol{u}$, T_e, T_i will be called the *hydrodynamical moments*. They are all defined as moments of the reference local equilibrium state, but they have an additional important property. *In their equations of evolution there is no contribution (or a negligibly small contribution) from the collision term.*

Among the remaining quantities defined as combinations of plasmadynamical moments we are left with the electric charge density σ and the electric current $\boldsymbol{j}$, i.e. typical *electrodynamical moments*. The former is a conserved quantity, but, as follows from the discussion in section 4.1, it plays a quite unimportant role in the class of slow motions considered here. It does not enter at all the hydrodynamical equations; it must only be kept in the Poisson equation. Thus, for the slow plasma motions, the condition of quasi-neutrality (1.13) prevails, and we may set $\sigma \simeq 0$.

The electric current $\boldsymbol{j}$ is a combination of plasmadynamical moments, but is *not* a conserved quantity. Its equation of evolution (3.4.30) contains a contribution from the collisions: the friction force $\boldsymbol{R}^{ei}$, which represents the collisional exchange of momentum between the two species. The same remarks are valid for the particle fluxes Γ^α, as can be seen from eq. (3.4.10).

The property of non-invariance under the collisions is shared by these plasmadynamical moments with the non-plasmadynamical moments. It is

therefore natural to group them together in a class called the *non-hydrodynamical moments*. In order to stress this common property, we adopt for j and for Γ^α a notation analogous to the non-plasmadynamical moments. For this purpose, we introduce dimensionless quantities,

$$j_r = en_e\left(\frac{T_e}{m_e}\right)^{1/2} h_r^{(1)}, \tag{4.4}$$

$$\Gamma_r^\alpha = n_\alpha\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} h_r^{\alpha(1)}. \tag{4.5}$$

The relation between these dimensionless quantities is easily derived:

$$h_r^{(1)} = -h_r^{e(1)} + \left(\frac{m_e}{T_e}\right)^{1/2} u_r. \tag{4.6}$$

We thus have a uniform system of notation for all the non-hydrodynamical moments: the letter h provided with appropriate sub- and superscripts. No confusion should arise from the fact that, in spite of the notation, $h_r^{(1)}$ and $h_r^{\alpha(1)}$ are *not* Hermitian moments (i.e. moments defined by the deviation from the local equilibrium). Indeed, the true Hermitian moment $h_r^{\alpha(1)}$, defined as the average of the Hermite polynomial $H_r^{(1)}$, is identically zero by constraint (3.16) and will therefore never again appear in the theory.

Among the non-hydrodynamical moments, there are a few which have an outstanding physical meaning. These are

- the *particle fluxes*: $\Gamma_r^\alpha \to h_r^{\alpha(1)}$,
- the *electric current*: $j_r \to h_r^{(1)}$,
- the *pressure tensors*: $\pi_{rs}^\alpha \to h_{rs}^{\alpha(2)}$,
- the *heat fluxes*: $q_r^\alpha \to h_r^{\alpha(3)}$.

It is important to note, – this will be proved in the next section – that these particular non-hydrodynamical moments, in addition to their exceptional physical importance, share a specific property. In their equations of evolution there appears a *source term which is a function of the hydrodynamical moments*. The equations for all the other non-hydrodynamical moments do not possess such a source term. Conversely, the moments of this class are *the only non-hydrodynamical quantities entering the equations for the hydrodynamical variables*, (3.4.21), (3.4.27), (3.4.28). These important properties justify the introduction of a subgroup of *privileged non-hydrodynamical moments*, containing $h_r^{\alpha(1)}$, $h_r^{(1)}$, $h_{rs}^{\alpha(2)}$ and $h_r^{\alpha(3)}$, and *only these*. In particular, there are *no*

Table 4.1
Classification of the moments. PD = Plasmadynamical; HD = Hydrodynamical

PD MOMENTS	T_e T_i n_e n_i		T_e T_i ρ ρu_r	HD MOMENTS
	u_r^e		σ	$\simeq 0$
	u_r^i		$h_r^{(1)} \leftarrow j_r$	Privileged
NON - PD MOMENTS	$h_{rs}^{\alpha(2)}$		$[\ h_r^{\alpha(1)} \leftarrow \Gamma_r^\alpha\ ]$ $h_{rs}^{\alpha(2)} \leftarrow \pi_{rs}^\alpha$	Privileged
	$h_r^{\alpha(3)}$		$h_r^{\alpha(3)} \leftarrow q_r^\alpha$	Privileged
	$h_{rs}^{\alpha(4)}$		$h_{rs}^{\alpha(4)}$	Non - Privileged
	$h_{rs}^{\alpha(6)}$		$h_{rs}^{\alpha(6)}$	Non - Privileged
	$h_r^{\alpha(5)}$		$h_r^{\alpha(5)}$	Non - Privileged
	$h_r^{\alpha(7)}$		$h_r^{\alpha(7)}$	Non - Privileged
	$h_{rsm}^{\alpha(3)}$		$h_{rsm}^{\alpha(3)}$	Non - Privileged
	$h_{rsmn}^{\alpha(4)}$		$h_{rsmn}^{\alpha(4)}$	Non - Privileged

NON - HD MOMENTS

scalar privileged moments; similarly, *there are no privileged moments among the third-order and higher-order anisotropies*. The difference in the behaviour of privileged and non-privileged moments will be discussed in chapter 5. It will also be shown in chapter 6 that the distinction between privileged and non-provileged moments has a deep thermodynamic basis.

The classification discussed in the present section is summarized in table 4.1.

4.5. *Equations of evolution for the moments. I. General form*

Having discussed the definition and the classification of the moments, we now proceed to establishing their equations of evolution. At this point we make the following remark.

The set of equations for the moments are often called the "fluid description" in the plasma physics literature. We believe that this name is ambiguous, for the following reason. If we write down the equations for *all* the moments, we would have to solve an *infinite hierarchy* of partial differential equations. Their solution is equivalent to the solution of the kinetic equation. In other words, the knowledge of all the moments is equivalent, by (3.11), to the knowledge of the distribution functions $f^{\alpha}(\boldsymbol{v}; \boldsymbol{x}, t)$.

One may also say that the complete set of moment equations is a *representation of the kinetic equation*. (This phrase has the same meaning as when we say that the Schrödinger equation for an anharmonic oscillator is equivalent to the infinite set of equations for the coefficients of the wave function, expanded in a series of Hermite polynomials). Thus, the complete set of moment equations is *not* a "fluid description", but rather, provides a bona fide microscopic description of the plasma.

However, in practice, the infinite hierarchy of moment equations cannot be solved. One therefore tries to justify an approximate truncation of the moment hierarchy at a certain level. When the level of truncation is reduced to the set of hydrodynamical moments, one is justified in speaking of a *fluid description* or a *hydrodynamical regime*. More precisely, in order to obtain a set of closed equations for the hydrodynamical moments, it is necessary to obtain *valid approximations of the privileged non-hydrodynamical moments as functionals of the hydrodynamical ones, evaluated at the same point in space and time*. This operation will be the object of the forthcoming chapters.

The derivation of the equations of evolution for the moments (or, briefly, *the moment equations*) is quite straightforward, although it requires a lot of labour and patience. The *hydrodynamical balance equations* were already derived in full generality in section 3.4. We now specialize them to the simpler case corresponding to the *local quasi-neutrality* constraint (1.8) and adopt the

dimensionless moment notation, by using eqs. (3.17), (3.18), (4.4) and (4.5) *.

The *continuity equation* (3.4.20) remains unchanged:

$$\partial_t \rho = -\nabla \cdot (\rho \boldsymbol{u}). \tag{5.1}$$

The *momentum balance equation* (3.4.21) becomes

$$\partial_t (\rho u_r) = -\nabla_s (\rho u_r u_s) - \nabla_r (n_e T_e + n_i T_i) - \sqrt{2}\, \nabla_s \left(n_e T_e h_{rs}^{e(2)} + n_i T_i h_{rs}^{i(2)} \right) + \frac{e}{c} n_e \left(\frac{T_e}{m_e} \right)^{1/2} \varepsilon_{rsm} h_s^{(1)} B_m. \tag{5.2}$$

The *temperature equations* (3.4.27), (3.4.28) become

$$\partial_t T_e = -\boldsymbol{u} \cdot \nabla T_e - \tfrac{2}{3} T_e \nabla \cdot \boldsymbol{u} + \left(\frac{T_e}{m_e} \right)^{1/2} h_r^{(1)} \nabla_r T_e + \tfrac{2}{3} T_e \nabla_r \left[\left(\frac{T_e}{m_e} \right)^{1/2} h_r^{(1)} \right] - \tfrac{2}{3} \sqrt{2}\, T_e h_{rs}^{e(2)} \nabla_r u_s + \tfrac{2}{3} \sqrt{2}\, T_e h_{rs}^{e(2)} \nabla_r \left[\left(\frac{T_e}{m_e} \right)^{1/2} h_s^{(1)} \right] - \frac{\sqrt{10}}{3 n_e} \nabla_r \left[n_e T_e \left(\frac{T_e}{m_e} \right)^{1/2} h_r^{e(3)} \right] - \frac{1}{Z} Q^{(2)} - \tfrac{2}{3} T_e h_n^{(1)} Q_n^{(1)}, \tag{5.3}$$

and

$$\partial_t T_i = -\boldsymbol{u} \cdot \nabla T_i - \tfrac{2}{3} T_i \nabla \cdot \boldsymbol{u} - \tfrac{2}{3} \sqrt{2}\, T_i h_{rs}^{i(2)} \nabla_r u_s - \frac{\sqrt{10}}{3 n_i} \nabla_r \left[n_i T_i \left(\frac{T_i}{m_i} \right)^{1/2} h_r^{i(3)} \right] + Q^{(2)}. \tag{5.4}$$

* Here and in the forthcoming equations we shall use a "mixed notation system": On the right-hand side of the equations we use the two-fluid variables n_e, n_i which lead to more compact and more transparent equations, rather than translating them explicitly into one-fluid variables by using table 3.2.2, column C, i.e. $n_e = Z\rho/m_i$, $n_i = \rho/m_i$. For instance, the second term on the right-hand side of (5.2), when written consistently in terms of one-fluid variables, would be $-(Z/m_i)\, \nabla_r [\rho (T_e + Z^{-1} T_i)]$.

The collisional terms appearing here are related to the quantities $Q^{\alpha}(=Q^{\alpha\alpha'})$ defined in (3.4.12), (3.4.13) as

$$Q^{\mathrm{ie}} = \tfrac{3}{2} n_{\mathrm{i}} Q^{(2)}, \tag{5.5}$$

$$Q^{\mathrm{ei}} = \tfrac{3}{2} n_{\mathrm{e}} Q^{\mathrm{ei}(2)}. \tag{5.6}$$

Introducing also the reduced friction force from (3.4.9) and (3.4.13) which is scaled, for convenience, as

$$R_r^{\mathrm{ei}} = -n_{\mathrm{e}} m_{\mathrm{e}} \left(\frac{T_{\mathrm{e}}}{m_{\mathrm{e}}} \right)^{1/2} Q_r^{(1)}, \tag{5.7}$$

we easily find that (3.4.15) implies:

$$Q^{\mathrm{ei}(2)} = -\frac{1}{Z} Q^{(2)} - \tfrac{2}{3} T_{\mathrm{e}} h_n^{(1)} Q_n^{(1)}. \tag{5.8}$$

The collisional terms $Q_r^{(1)}$ and $Q^{(2)}$ will be evaluated in the next section. It will turn out that $Q^{(2)}$ is a term of order μ $(= m_{\mathrm{e}}/m_{\mathrm{i}})$, hence negligible. Thus, the only collisional contribution to (5.3) and (5.4) is the last term of the former, which is non-linear in the non-hydrodynamical moments, hence it is usually small. These facts justify our treatment of the *two* temperatures as "quasi-conserved" quantities.

The structure of the hydrodynamical balance equations appears now very clearly. Their right-hand side contains two types of terms. The first group involves only hydrodynamical quantities. The second group involves the *privileged* non-hydrodynamical moments $h_r^{(1)}$, $h_{rs}^{\alpha(2)}$, $h_r^{\alpha(3)}$. Equations (5.3) and (5.4) also involve a collisional contribution, which is very small.

For completeness we also rewrite the *Maxwell equations* in the new notation:

$$\nabla \cdot \boldsymbol{E} = 4\pi\sigma (\simeq 0), \qquad \nabla \wedge \boldsymbol{E} = -\frac{1}{c} \partial_t \boldsymbol{B},$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \wedge \boldsymbol{B} = \frac{4\pi e}{c} n_{\mathrm{e}} \left(\frac{T_{\mathrm{e}}}{m_{\mathrm{e}}} \right)^{1/2} h_r^{(1)}. \tag{5.9}$$

We now turn to the *non-hydrodynamical moment equations*. We begin with the electrodynamical moment equations, which were already derived in section 3.4, but which must now be transformed into equations for the dimensionless

moments. In this process, we note that $\partial_t j_r$ contains two terms: one involving $\partial_t h_r^{(1)}$ and the other involving $\partial_t(n_e T_e^{1/2})$. The latter must be made explicit by using the hydrodynamical equations (5.1)–(5.4) (with $Q^{(2)} \approx 0$). In this and the following calculations, the following lemma is very useful:

$$\partial_t \left(n_\alpha T_\alpha^{p/2}\right)^{-1} = \left(n_\alpha T_\alpha^{p/2}\right)^{-1}$$

$$\times \left\{ \left(1 + \frac{p}{3}\right) \nabla_n \left[u_n - \delta^{\alpha e} \left(\frac{T_e}{m_e}\right)^{1/2} h_n^{(1)} \right] \right.$$

$$+ \left[u_n - \delta^{\alpha e} \left(\frac{T_e}{m_e}\right)^{1/2} h_n^{(1)} \right] \frac{1}{n_\alpha T_\alpha^{p/2}} \nabla_n \left(n_\alpha T_\alpha^{p/2}\right)$$

$$+ \frac{p\sqrt{2}}{3} h_{mn}^{\alpha(2)} \nabla_m \left[u_n - \delta^{\alpha e} \left(\frac{T_e}{m_e}\right)^{1/2} h_n^{(1)} \right]$$

$$\left. + \frac{p\sqrt{10}}{6 n_\alpha T_\alpha} \nabla_n \left[n_\alpha T_\alpha \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} h_n^{\alpha(3)} \right] - \delta^{\alpha e} \frac{p}{3} h_n^{(1)} Q_n^{(1)} \right\}, \tag{5.10}$$

where p is an arbitrary integer.

We have seen that the charge conservation equation (3.4.29) reduces, for slow motions, to a constraint on the current (1.10) or, with (4.4),

$$\nabla_n \left(n_e T_e^{1/2} h_n^{(1)}\right) = 0. \tag{5.11}$$

We now combine (3.4.30), (1.13), (1.15), (4.4), (5.7) and (5.11), in order to obtain the *generalized Ohm law* for the dimensionless current density,

$$\partial_t h_r^{(1)} = \left(\frac{m_e}{T_e}\right)^{1/2} \left(\frac{e}{m_e} E_r + \frac{1}{m_e n_e} \nabla_r (n_e T_e) + \frac{e}{m_e c} \varepsilon_{rmn} u_m B_n \right)$$

$$- \frac{e}{m_e c} \varepsilon_{rmn} h_m^{(1)} B_n + Q_r^{(1)} + U_r^{(1)} + C_r^{(1)} + N_r^{(1)}. \tag{5.12}$$

Here $Q_r^{(1)}$ is the dimensionless friction force (5.7), which will be evaluated in section 4.6. The remaining terms are

$$U_r^{(1)} = \sqrt{2}\left(\frac{T_e}{m_e}\right)^{1/2} \frac{1}{n_e T_e} \nabla_s \left[n_e \left(T_e h_{rs}^{e(2)} - \mu T_i h_{rs}^{i(2)} \right) \right], \tag{5.13}$$

$$C_r^{(1)} = -\boldsymbol{u} \cdot \nabla h_r^{(1)} - h_m^{(1)} \nabla_m u_r + \tfrac{1}{3} h_r^{(1)} \nabla \cdot \boldsymbol{u} + \frac{\sqrt{2}}{3} h_r^{(1)} h_{mn}^{e(2)} \nabla_m \left[u_n - \left(\frac{T_e}{m_e}\right)^{1/2} h_n^{(1)} \right], \tag{5.14}$$

$$N_r^{(1)} = \left(\frac{T_e}{m_e}\right)^{1/2} \left\{ \frac{\sqrt{10}}{6} h_r^{(1)} \frac{1}{n_e T_e^{3/2}} \nabla_m \left(n_e T_e^{3/2} h_m^{e(3)} \right) + h_m^{(1)} \nabla_m h_r^{(1)} - \tfrac{1}{3} h_r^{(1)} T_e^{-1/2} \nabla_m \left(T_e^{1/2} h_m^{(1)} \right) \right\} - \tfrac{1}{3} h_r^{(1)} h_n^{(1)} Q_n^{(1)}. \tag{5.15}$$

These terms will be discussed below.

In the alternative description (4.3), in which one of the particle fluxes replaces the electric current, we obtain from (3.4.10)

$$\partial_t h_r^{\alpha(1)} = \left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \left[-\frac{1}{m_\alpha n_\alpha} \nabla_r (n_\alpha T_\alpha) + \frac{e_\alpha}{m_\alpha} E_r \right] + \frac{e_\alpha}{m_\alpha c} \varepsilon_{rmn} h_m^{\alpha(1)} B_n + Q_r^{\alpha(1)} + U_r^{\alpha(1)} + N_r^{\alpha(1)}. \tag{5.16}$$

The friction term for species α is defined as

$$Q_r^{\alpha(1)} = n_\alpha m_\alpha \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} R_r^\alpha. \tag{5.17}$$

From eqs. (3.4.14) and (5.7) we obtain

$$Q_r^{e(1)} = -Q_r^{(1)}$$

$$Q_r^{i(1)} = \frac{1}{Z\mu^{1/2}} \left(\frac{T_e}{T_i}\right)^{1/2} Q_r^{(1)}. \tag{5.18}$$

The other terms in eq. (5.16) are

$$U_r^{\alpha(1)} = -\sqrt{2}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \frac{1}{n_\alpha T_\alpha} \nabla_s \left(n_\alpha T_\alpha h_{rs}^{\alpha(2)}\right), \tag{5.19}$$

$$\begin{aligned} N_r^{\alpha(1)} = \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \Bigg[& -\frac{1}{n_\alpha T_\alpha} \nabla_s \left(n_\alpha T_\alpha h_r^{\alpha(1)} h_s^{\alpha(1)}\right) \\ & + \frac{4}{3T_\alpha^{1/2}} h_r^{\alpha(1)} \nabla_n \left(T_\alpha^{1/2} h_n^{\alpha(1)}\right) \\ & + \frac{1}{n_\alpha T_\alpha^{1/2}} h_r^{\alpha(1)} h_n^{\alpha(1)} \nabla_n \left(n_\alpha T_\alpha^{1/2}\right) \\ & + \tfrac{1}{3}\sqrt{2}\, T_\alpha^{-1/2} h_r^{\alpha(1)} h_{mn}^{\alpha(2)} \nabla_n \left(T_\alpha^{1/2} h_n^{\alpha(1)}\right) \\ & + \frac{\sqrt{10}}{6n_\alpha T_\alpha^{3/2}} h_r^{\alpha(1)} \nabla_n \left(n_\alpha T_\alpha^{3/2} h_n^{\alpha(3)}\right) \Bigg] \\ & - \tfrac{1}{3} h_r^{(1)} h_n^{(1)} Q_n^{(1)} \delta^{\alpha e}. \end{aligned} \tag{5.20}$$

The derivation of the *Hermitian moment equations* is not difficult, but requires some care. As an example containing all the delicate points of the calculation, we sketch in some detail the derivation of the equation for $h_r^{\alpha(3)}$ in Appendix A1. Those hints are sufficient for enabling the reader to understand or check the forthcoming equations. The collisional contributions in all these equations will be evaluated in section 4.6; at present, we simply introduce the definition

$$Q_{r_1 r_2 \ldots}^{\alpha(m)} = n_\alpha^{-1} \int \mathrm{d}\boldsymbol{v}\, H_{r_1 r_2 \ldots}^{(m)} \left((m_\alpha/T_\alpha)^{1/2} (\boldsymbol{v} - \boldsymbol{u}^\alpha)\right) \mathscr{K}^\alpha. \tag{5.21}$$

We now present the first equations of the Hermitian moment hierarchy. We actually truncate the hierarchy at the level of the 29 M approximation (3.22). Clearly, the 21 M approximation is recovered by setting everywhere $h_{rs}^{\alpha(6)} = h_r^{\alpha(7)} = 0$; the 13 M approximation is obtained by setting also $h_{rs}^{\alpha(4)} = h_r^{\alpha(5)} = 0$. The equations are presented all together. Their common structure and the meaning of the notation is discussed below.

(A) Vector moments

$$\partial_t h_r^{\alpha(3)} = -\sqrt{\frac{5}{2}}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\frac{1}{T_\alpha}\nabla_r T_\alpha + \frac{e_\alpha}{m_\alpha c}\varepsilon_{rmn}h_m^{\alpha(3)}B_n + Q_r^{\alpha(3)} + U_r^{\alpha(3)} + D_r^{\alpha(3)} + C_r^{\alpha(3)} + N_r^{\alpha(3)}, \tag{5.22}$$

$$\partial_t h_r^{\alpha(5)} = \frac{e_\alpha}{m_\alpha c}\varepsilon_{rmn}h_m^{\alpha(5)}B_n + Q_r^{\alpha(5)} + U_r^{\alpha(5)} + D_r^{\alpha(5)} + C_r^{\alpha(5)} + N_r^{\alpha(5)}, \tag{5.23}$$

$$\partial_t h_r^{\alpha(7)} = \frac{e_\alpha}{m_\alpha c}\varepsilon_{rmn}h_m^{\alpha(7)}B_n + Q_r^{\alpha(7)} + D_r^{\alpha(7)} + C_r^{\alpha(7)} + N_r^{\alpha(7)}. \tag{5.24}$$

(B) Tensor moments.

$$\partial_t h_{rs}^{\alpha(2)} = -\sqrt{2}\,\mathscr{T}_{rs|pq}\ \nabla_p u_q + \frac{2e_\alpha}{m_\alpha c}\mathscr{T}_{rs|pq}\ \varepsilon_{pmn}h_{qm}^{\alpha(2)}B_n + Q_{rs}^{\alpha(2)} + U_{rs}^{\alpha(2)} + D_{rs}^{\alpha(2)} + C_{rs}^{\alpha(2)} + N_{rs}^{\alpha(2)}, \tag{5.25}$$

$$\partial_t h_{rs}^{\alpha(4)} = \frac{2e_\alpha}{m_\alpha c}\mathscr{T}_{rs|pq}\ \varepsilon_{pmn}h_{qm}^{\alpha(4)}B_n + Q_{rs}^{\alpha(4)} + U_{rs}^{\alpha(4)} + D_{rs}^{\alpha(4)} + C_{rs}^{\alpha(4)} + N_{rs}^{\alpha(4)}, \tag{5.26}$$

$$\partial_t h_{rs}^{\alpha(6)} = \frac{2e_\alpha}{m_\alpha c}\mathscr{T}_{rs|pq}\ \varepsilon_{pmn}h_{qm}^{\alpha(6)}B_n + Q_{rs}^{\alpha(6)} + U_{rs}^{\alpha(6)} + D_{rs}^{\alpha(6)} + C_{rs}^{\alpha(6)} + N_{rs}^{\alpha(6)}. \tag{5.27}$$

In the tensorial equations we have introduced the convenient *symmetrization operator* $\mathscr{T}_{rs|pq}$ defined as

$$\mathscr{T}_{rs|pq} = \tfrac{1}{2}\left(\delta_{rp}\delta_{sq} + \delta_{rq}\delta_{sp} - \tfrac{1}{3}\delta_{rs}\delta_{pq}\right). \tag{5.28}$$

Although the detailed form of the moment equations is quite complicated, it is important to analyze the common structure of eqs. (5.12), (5.16), (5.22)–(5.27). The right-hand side of the equation for the moment of order n features in succession:

(1) A term involving only *the gradients of the hydrodynamical variables* ρ, $\boldsymbol{u}$, T_α *and the electromagnetic fields* $\boldsymbol{E}$ and $\boldsymbol{B}$: it will be called the *source term*. It is because of the presence of these source terms that the non-hydrodynamical moments can ultimately be expressed in terms of the hydrodynamical ones, and that the hydrodynamical equations can be closed. It is important to note that *the source term is different from zero only in the equations for* $h_r^{(1)}$, $h_r^{\alpha(1)}$, $h_{rs}^{\alpha(2)}$ and $h_r^{\alpha(3)}$. This confirms the statement in section 4.4 by which these moments are given a *privileged* role. This property has important consequences.

(2) A *magnetic term* originating from the Lorentz force. It is always proportional to the vector product of the moment of order n and the magnetic field. This term will, of course, play a predominant role in the study of magnetically confined plasmas.

(3) A *collisional term* $Q_{\ldots}^{\alpha(n)}$ is always present, because the non-hydrodynamical moments are averages of dynamical variables which are *not* collisional invariants. These terms will be called collectively: the *generalized friction terms*. They contain both linear and non-linear contributions, and will be evaluated in section 4.6.

(4) A "hierarchical" term, or "*up-term*" $U_{\ldots}^{\alpha(n)}$ which contains the moment of order $(n+1)$ produced by the flow term of the kinetic equation. We have given $U_r^{(1)}$ in (5.13) and $U_r^{\alpha(1)}$ in (5.19); we give below the form of $U_{rs}^{\alpha(2)}$, $U_r^{\alpha(3)}$ and $U_r^{\alpha(5)}$, but omit the others, because they will not be used later. Note that there is no up-term in (5.24), because $h_{rs}^{\alpha(8)} = 0$ in the 29 M approximation.

$$U_{rs}^{\alpha(2)} = -\frac{2}{\sqrt{5}} \mathscr{T}_{rs\,[pq}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \frac{1}{n_\alpha T_\alpha^{3/2}} \nabla_p \left(n_\alpha T_\alpha^{3/2} h_{q]}^{\alpha(3)}\right), \tag{5.29}$$

$$U_r^{\alpha(3)} = -\left(\frac{14 T_\alpha}{5 m_\alpha}\right)^{1/2} \frac{1}{n_\alpha T_\alpha^2} \nabla_m \left(n_\alpha T_\alpha^2 h_{rm}^{\alpha(4)}\right), \tag{5.30}$$

$$U_r^{\alpha(5)} = -\frac{6}{\sqrt{10}}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \frac{1}{n_\alpha T_\alpha^3} \nabla_m \left(n_\alpha T_\alpha^3 h_{rm}^{\alpha(6)}\right). \tag{5.31}$$

(5) A "*down-term*" $D_{\ldots}^{\alpha(n)}$ which involves the spatial derivative of the lower moment of order $(n-1)$. This term is, of course, absent in the equation for

$h_r^{(1)}$ and for $h_r^{\alpha(1)}$. For the other relevant moments it is given by

$$D_{rs}^{\alpha(2)} = \sqrt{2}\,\delta^{\alpha e}\mathscr{T}_{rs|pq}\left(\frac{T_e}{m_e}\right)^{1/2}\nabla_p h_q^{(1)}, \tag{5.32}$$

$$D_r^{\alpha(3)} = -\frac{2}{\sqrt{5}}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} T_\alpha^{-7/2}\,\nabla_m\left(T_\alpha^{7/2}h_{rs}^{\alpha(2)}\right), \tag{5.33}$$

$$D_r^{\alpha(5)} = -\sqrt{\frac{2}{5}}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\left[2T_\alpha^{-11/2}\,\nabla_m\left(T_\alpha^{11/2}h_{rm}^{\alpha(4)}\right)\right.$$

$$\left. + \sqrt{14}\,h_{rm}^{\alpha(2)}T_\alpha^{-1}\,\nabla_m T_\alpha\right]. \tag{5.34}$$

(6) A group of *convective terms* $C_{\cdots}^{\alpha(n)}$ involving the moment of order n (and possibly others of lower order) and the velocities $\boldsymbol{u}^\alpha$, as well as their spatial derivatives (see 5.14). Note that in eq. (5.16), because of the nature of $h_r^{\alpha(1)} \sim u_r^\alpha$, these terms are absorbed in the nonlinear terms $N_r^{\alpha(1)}$. We quote only the explicit forms for the privileged moments,

$$C_{rs}^{\alpha(2)} = -\boldsymbol{u}^\alpha\cdot\nabla h_{rs}^{\alpha(2)} + \tfrac{2}{3}h_{rs}^{\alpha(2)}\,\nabla\cdot\boldsymbol{u}^\alpha - 2\mathscr{T}_{rs|pq}h_{pm}^{\alpha(2)}\,\nabla_m u_q^\alpha$$

$$+ \tfrac{2}{3}\sqrt{2}\,h_{rs}^{\alpha(2)}h_{mn}^{\alpha(2)}\,\nabla_n u_m^\alpha, \tag{5.35}$$

$$C_r^{\alpha(3)} = -\boldsymbol{u}^\alpha\cdot\nabla h_r^{\alpha(3)} - \tfrac{7}{5}h_m^{\alpha(3)}\,\nabla_m u_r^\alpha - \tfrac{2}{5}h_m^{\alpha(3)}\,\nabla_r u_m^\alpha$$

$$+ \tfrac{3}{5}h_r^{\alpha(3)}\,\nabla\cdot\boldsymbol{u}^\alpha + \sqrt{2}\,h_r^{\alpha(3)}h_{ns}^{\alpha(2)}\,\nabla_n u_s^\alpha. \tag{5.36}$$

(7) Finally, there is a set of complicated *non-linear terms* $N_{\cdots}^{\alpha(n)}$ in addition to the non-linear contributions contained in the generalized friction terms (see eqs. 5.15 and 5.20),

$$N_{rs}^{\alpha(2)} = \frac{\sqrt{10}}{3}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\frac{1}{n_\alpha T_\alpha^{3/2}}h_{rs}^{\alpha(2)}\,\nabla_m\left(n_\alpha T_\alpha^{3/2}h_n^{\alpha(3)}\right) - \tfrac{2}{3}\delta^{\alpha e}h_{rs}^{e(2)}h_n^{(1)}Q_n^{(1)}, \tag{5.37}$$

$$N_r^{\alpha(3)} = \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\left[\sqrt{\frac{5}{2}}\,\frac{1}{n_\alpha T_\alpha^{3/2}} h_r^{\alpha(3)}\,\nabla_p\left(n_\alpha T_\alpha^{3/2} h_p^{\alpha(3)}\right)\right.$$

$$\left. + \sqrt{\frac{8}{5}}\,\frac{1}{n_\alpha T_\alpha} h_{rm}^{\alpha(2)}\,\nabla_p\left(n_\alpha T_\alpha h_{mp}^{\alpha(2)}\right)\right]$$

$$+ \left(2/\sqrt{5}\right)\left(\delta^{\alpha e} + Z\mu^{1/2}\left(T_e/T_i\right)^{1/2}\delta^{\alpha i}\right) h_{rm}^{\alpha(2)} Q_m^{(1)} - \delta^{\alpha e} h_r^{e(3)} h_m^{(1)} Q_m^{(1)}. \tag{5.38}$$

(8) We finally underscore the fact that the *electric field* $\boldsymbol{E}$ only appears in the source term for the first order moments $h_r^{(1)}$, $h_r^{\alpha(1)}$.

4.6. *Equations of evolution for the moments. II. The generalized frictions*

In the previous section we left unspecified the contribution of the collision term to the Hermitian moment equations, i.e. the *generalized frictions*. We now turn to the explicit evaluation of these quantities. A typical generalized friction is made up of two terms, corresponding to like-particle and to unlike-particle collisions:

$$Q_\psi^\alpha = Q_\psi^{\alpha\alpha} + Q_\psi^{\alpha\alpha'} \tag{6.1}$$

(where, as usual, α' is the complement of α). $Q_\psi^{\alpha\beta}$ is the contribution of the $(\alpha\beta)$ collisions to the rate of change of the average of a dynamical function ψ of the dimensionless variable $\boldsymbol{c}$, defined by (3.4),

$$Q_\psi^{\alpha\beta} = \frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\; \psi\left(\left(m_\alpha/T_\alpha\right)^{1/2}\left(\boldsymbol{v} - \boldsymbol{u}^\alpha\right)\right)\mathscr{X}^{\alpha\beta}. \tag{6.2}$$

We start with the discussion of the *electron–electron collisions*. Using (3.3.8) and (3.8), and performing an integration by parts, we find

$$Q_\psi^{ee} = -\frac{2\pi e^4 \ln\Lambda}{m_e^2 n_e}\int \mathrm{d}\boldsymbol{v}_1 \int \mathrm{d}\boldsymbol{v}_2 \left\{\partial_{1m}\psi\left[\left(\frac{m_e}{T_e}\right)^{1/2}\left(\boldsymbol{v}_1 - \boldsymbol{u}^e\right)\right]\right\}$$

$$\times G_{mn}(\boldsymbol{g})\left(\partial_{1n} - \partial_{2n}\right) f^e(\boldsymbol{v}_1)\, f^e(\boldsymbol{v}_2)$$

$$= -\frac{2\pi e^4 \ln\Lambda}{m_e^2 n_e} n_e^2 \left(\frac{m_e}{T_e}\right)^{3/2} \int \mathrm{d}\boldsymbol{c}_1 \int \mathrm{d}\boldsymbol{c}_2 \left[\frac{\partial\psi(\boldsymbol{c}_1)}{\partial c_{1m}}\right] G_{mn}(\gamma)$$

$$\times\left(\frac{\partial}{\partial c_{1n}} - \frac{\partial}{\partial c_{2n}}\right)\phi^0(c_1)\,\phi^0(c_2)\,(1+\chi^e(\boldsymbol{c}_1))(1+\chi^e(\boldsymbol{c}_2)),$$

which can be rewritten as

$$Q_\psi^{ee} = -\frac{3\sqrt{2\pi}}{4}\frac{n_e}{Z^2 n_i}\frac{1}{\tau_e}\,\mathbb{Q}_\psi^{ee}, \tag{6.3}$$

where we introduce the characteristic time τ_e, which is a good measure of the *electron relaxation time*,

$$\frac{1}{\tau_e} = \frac{4\sqrt{2\pi}}{3}\frac{Z^2 e^4 n_i \ln \Lambda}{m_e^{1/2} T_e^{3/2}}. \tag{6.4}$$

We recall here definition (**2**.6.27) of the *Coulomb logarithm*:

$$\ln \Lambda = \ln \frac{\frac{3}{2}(T_e + T_i)\lambda_D}{Ze^2}, \tag{6.5}$$

where the Debye length λ_D is given in (**2**.4.8). The dimensionless quantity $\mathbb{Q}_\psi^{ee}$ is defined as

$$\mathbb{Q}_\psi^{ee} = \int d\boldsymbol{c}_1\, d\boldsymbol{c}_2 \left(\frac{\partial \psi(\boldsymbol{c}_1)}{\partial c_{1m}}\right) G_{mn}(\boldsymbol{\gamma}) \left(\frac{\partial}{\partial c_{1n}} - \frac{\partial}{\partial c_{2n}}\right) \phi^e(\boldsymbol{c}_1)\, \phi^e(\boldsymbol{c}_2), \tag{6.6}$$

where $\boldsymbol{\gamma}$ is the dimensionless relative velocity,

$$\boldsymbol{\gamma} = \boldsymbol{c}_1 - \boldsymbol{c}_2.$$

The Landau tensor $G_{mn}(\boldsymbol{\gamma})$ (see eq. **2**.6.26) is evaluated as a function of the dimensionless relative velocity,

$$G_{mn}(\boldsymbol{\gamma}) = \frac{\gamma^2 \delta_{mn} - \gamma_m \gamma_n}{\gamma^3}. \tag{6.7}$$

$\phi^\alpha(\boldsymbol{c})$ denotes the (perturbed) dimensionless distribution function,

$$\phi^\alpha(\boldsymbol{c}) = \phi^0(c)[1 + \chi^\alpha(\boldsymbol{c})]. \tag{6.8}$$

A similar calculation yields the following expression of the *ion–ion generalized friction*:

$$Q_\psi^{ii} = -\frac{3\sqrt{2\pi}}{4}\frac{1}{\tau_i}\,\mathbb{Q}_\psi^{ii}, \tag{6.9}$$

where $\mathbb{Q}_\psi^{ii}$ is defined by the same dimensionless function (6.6), in which all the subscripts e are replaced by i. We introduce here the *ion relaxation time* τ_i, defined as *

$$\frac{1}{\tau_i} = \frac{4\sqrt{2\pi}}{3} \frac{Z^4 e^4 n_i \ln \Lambda}{m_i^{1/2} T_i^{3/2}}. \tag{6.10}$$

Hence

$$\tau_i = \mu^{-1/2} Z^{-2} (T_i/T_e)^{3/2} \tau_e. \tag{6.11}$$

For the *electron–ion collisions* we use the *Lorentz form* of the collision operator, described in section 2.8, eqs. (2.8.1), (2.8.11). By means of several partial integrations we obtain

$$\begin{aligned} Q_\psi^{ei} &\equiv \frac{1}{n_e} \int d\boldsymbol{v}_1 \, \psi\big((m_e/T_e)^{1/2}(\boldsymbol{v}_1 - \boldsymbol{u}^e)\big) \mathscr{K}^{ei} \\ &= -\frac{3\sqrt{2\pi}}{4} \frac{1}{\tau_e} \mathbb{Q}_\psi^{ei} \end{aligned} \tag{6.12}$$

with

$$\mathbb{Q}_\psi^{ei} = \int d\boldsymbol{c} \, G_{jk}(\boldsymbol{c}) \, \mathscr{H}_{jk}^e(\boldsymbol{c}), \tag{6.13}$$

which involves the Landau tensor of the dimensionless variable $\boldsymbol{c}$, and the following functional of the dimensionless distribution function ϕ_j^e:

$$\begin{aligned} \mathscr{H}_{jk}^e = {} & \frac{\partial \psi(\boldsymbol{c})}{\partial c_j} \frac{\partial \phi^e(\boldsymbol{c})}{\partial c_k} + h_n^{(1)} \frac{\partial}{\partial c_n} \left(\frac{\partial \psi(\boldsymbol{c})}{\partial c_j} \frac{\partial \phi^e}{\partial c_k} \right) \\ & + \tfrac{1}{2} \tilde{\alpha}_{mn} \frac{\partial^2}{\partial c_m \, \partial c_n} \left(\frac{\partial \psi(\boldsymbol{c})}{\partial c_j} \frac{\partial \phi^e(\boldsymbol{c})}{\partial c_k} \right) + \cdots \\ & + \mu \left[\frac{\partial}{\partial c_k} \left(\phi^e(\boldsymbol{c}) \frac{\partial \psi(\boldsymbol{c})}{\partial c_j} \right) \right] + \cdots . \end{aligned} \tag{6.14}$$

* The *ion* relaxation time τ_i as defined here differs by a factor $\sqrt{2}$ from Braginskii's (1965) definition [his formula (2.5i) has a factor $\sqrt{\pi}$ instead of $\sqrt{2\pi}$ in our eq. (6.10)]. We do not see any reason for introducing such an artificial asymmetry between the electrons and the ions.

In deriving this equation, we made use of the relation

$$\boldsymbol{u}^{\mathrm{ei}} \equiv \boldsymbol{u}^{\mathrm{e}} - \boldsymbol{u}^{\mathrm{i}} = -\frac{1}{en_{\mathrm{e}}}\boldsymbol{j}, \tag{6.15}$$

valid for a quasi-neutral plasma, as well as of (4.4). The dimensionless tensor $\tilde{\alpha}_{mn}$ is easily obtained from (2.8.12):

$$\tilde{\alpha}_{mn} = \mu\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\left(\delta_{mn} + h^{\mathrm{i}(2)}_{mn}\right) + h^{(1)}_{m}h^{(1)}_{n}. \tag{6.16}$$

Finally, the *ion–electron generalized friction* is written in the following form, using (2.8.13),

$$Q^{\mathrm{ie}}_{\psi} \equiv \frac{1}{n_{\mathrm{i}}}\int \mathrm{d}\boldsymbol{v}\ \psi\left((m_{\mathrm{i}}/T_{\mathrm{i}})^{1/2}(\boldsymbol{v}-\boldsymbol{u}^{i})\right)\mathscr{K}^{\mathrm{ie}}$$

$$= -\frac{1}{\tau_{\mathrm{i}}}\frac{n_{\mathrm{e}}}{Z^2 n_{\mathrm{i}}}\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\,\mathbb{Q}^{\mathrm{ie}}_{\psi}, \tag{6.17}$$

with

$$\mathbb{Q}^{\mathrm{ie}}_{\psi} = \int \mathrm{d}\boldsymbol{c}\frac{\partial\psi(\boldsymbol{c})}{\partial c_n}\tilde{\mathscr{J}}^{\mathrm{ie}}_{n}(\boldsymbol{c}). \tag{6.18}$$

The dimensionless vector $\tilde{\mathscr{J}}^{\mathrm{ie}}_{n}$ is obtained from (2.8.17):

$$\tilde{\mathscr{J}}^{\mathrm{ie}}_{n}(\boldsymbol{c}) = -L_n\,\phi^{\mathrm{i}}(\boldsymbol{c}) + M_{nm}k_m\,\phi^{\mathrm{i}}(\boldsymbol{c}) - \tfrac{1}{2}N_{nmp}k_m k_p\,\phi^{\mathrm{i}}(\boldsymbol{c})$$

$$+\sqrt{\mu}\left(\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)^{-1/2}\left(\hat{L}_{nm}\frac{\partial\phi^{\mathrm{i}}(\boldsymbol{c})}{\partial c_m} + \cdots\right), \tag{6.19}$$

with

$$k_m = \sqrt{\mu}\left(\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)^{1/2} c_m + h^{(1)}_{m}.$$

The dimensionless coefficients in (6.19) are readily obtained from (2.8.18):

$$L_n = \frac{m_e}{T_e} \int \mathrm{d}\boldsymbol{c}\, G_{nr}(\boldsymbol{c}) \frac{\partial \phi^e(\boldsymbol{c})}{\partial c_r}$$

$$\hat{L}_{nm} = \left(\frac{m_e}{T_e}\right)^{1/2} \int \mathrm{d}\boldsymbol{c}\, G_{nm}(\boldsymbol{c})\, \phi^e(\boldsymbol{c}) \tag{6.20}$$

and so forth.

After these transformations, the problem is reduced to the evaluation of the four quantities $\mathbb{Q}^{\alpha\beta}$ and of the coefficients of (6.20). All of these operations involve integrals over a linear or a quadratic function of ϕ^α. In the Hermitian moment method, one simply substitutes one of the forms (3.20)–(3.22), and obtains an explicit function of the velocity, enabling one to perform the integrations. We now sketch the integration procedure.

Remember that the functions $\psi(\boldsymbol{c})$ of interest are polynomials in the components of $\boldsymbol{c}$. The simplest of all the integrals on our list are those of (6.18). If the coefficients L, M,... are known, the integrals to be calculated are of the form

$$\int \mathrm{d}\boldsymbol{c} \exp(-c^2/2) c^p c_m c_n c_r \ldots, \tag{6.21}$$

i.e. they are simply moments of a Gaussian distribution.

Next, we treat the integrals appearing in $\mathbb{Q}^{ei}_{\psi}$, (6.13) and in the coefficients of (6.20). It is clear from (6.14) that $\mathscr{H}^e_{jk}(\boldsymbol{c})$ has the form of a Gaussian, $\exp(-c^2/2)$, multiplied by a polynomial; hence, both (6.13) and (6.20) reduce to a linear combination of integrals of the form

$$\int \mathrm{d}\boldsymbol{c} \exp(-c^2/2)\, G_{mn}(\boldsymbol{c})\, c^p c_q c_r c_s \ldots. \tag{6.22}$$

The contributions $\mathbb{Q}^{\alpha\alpha}$ of the like-particle collisions (6.5) require a more tedious calculation, because they involve two integrations, over $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, and because the Landau tensor mixes these two variables non-linearly. Therefore, after performing the differentiations, one goes over to the new integration variables $\boldsymbol{\gamma}$, $\boldsymbol{\Gamma}$,

$$\boldsymbol{\gamma} = \boldsymbol{c}_1 - \boldsymbol{c}_2, \qquad \boldsymbol{\Gamma} = \tfrac{1}{2}(\boldsymbol{c}_1 + \boldsymbol{c}_2),$$

$$\boldsymbol{c}_1 = \boldsymbol{\Gamma} + \tfrac{1}{2}\boldsymbol{\gamma}, \qquad \boldsymbol{c}_2 = \boldsymbol{\Gamma} - \tfrac{1}{2}\boldsymbol{\gamma},$$

$$\exp\left(-\tfrac{1}{2}(c_1^2 + c_2^2)\right) = \exp(-\Gamma^2 - \tfrac{1}{4}\gamma^2). \tag{6.23}$$

Table 6.1
Integrals involved in the calculation of the generalized frictions.

Gaussian moments

$$(2\pi)^{-3/2}\int \mathrm{d}\boldsymbol{\Gamma}\, \mathrm{e}^{-\Gamma^2}\, \Gamma^{2k} = \frac{3.5\cdots(2k+1)}{2^{k+(3/2)}}$$

$$(2\pi)^{-3/2}\int \mathrm{d}\boldsymbol{\Gamma}\, \mathrm{e}^{-\Gamma^2}\, \Gamma^{2k}\Gamma_m\Gamma_n = \delta_{mn}\frac{3.5.7\cdots(2k+3)}{3.2^{k+(5/2)}}$$

$$(2\pi)^{-3/2}\int \mathrm{d}\boldsymbol{\Gamma}\, \mathrm{e}^{-\Gamma^2}\, \Gamma^{2k}\Gamma_m\Gamma_n\Gamma_p\Gamma_q$$

$$= (\delta_{mn}\delta_{pq} + \delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np})\frac{5.7\cdots(2k+5)}{5.2^{k+(7/2)}}$$

$$(2\pi)^{-3/2}\int \mathrm{d}\boldsymbol{\Gamma}\, \mathrm{e}^{-\Gamma^2}\, \Gamma^{2k}\Gamma_m\Gamma_n\Gamma_p\Gamma_q\Gamma_r\Gamma_s$$

$$= \{\delta_{mn}(\delta_{pq}\delta_{rs} + \delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}) + \delta_{mp}(\delta_{nq}\delta_{rs} + \delta_{nr}\delta_{qs} + \delta_{ns}\delta_{qr})$$
$$+ \delta_{mq}(\delta_{np}\delta_{rs} + \delta_{nr}\delta_{ps} + \delta_{ns}\delta_{pr}) + \delta_{mr}(\delta_{np}\delta_{qs} + \delta_{nq}\delta_{ps} + \delta_{ns}\delta_{pq})$$
$$+ \delta_{ms}(\delta_{np}\delta_{qr} + \delta_{nq}\delta_{pr} + \delta_{nr}\delta_{pq})\}\frac{7.9\cdots(2k+7)}{7.2^{k+(9/2)}}$$

Integrals involving the Landau tensor

$$(2\pi)^{-3/2}\int \mathrm{d}\boldsymbol{c}\, \mathrm{e}^{-c^2/2}\, G_{rs}(\boldsymbol{c})\, c^{2k} = \delta_{rs}\frac{2^{k+1}k!}{3\sqrt{2\pi}}$$

$$(2\pi)^{-3/2}\int \mathrm{d}\boldsymbol{c}\, \mathrm{e}^{-c^2/2}\, G_{rs}(\boldsymbol{c})\, c^{2k}c_mc_n$$

$$= (4\delta_{rs}\delta_{mn} - \delta_{rm}\delta_{sn} - \delta_{rn}\delta_{sm})\frac{2^{k+2}(k+1)!}{15\sqrt{2\pi}}$$

$$(2\pi)^{-3/2}\int \mathrm{d}\boldsymbol{c}\, \mathrm{e}^{-c^2/2}\, G_{rs}(\boldsymbol{c})\, c^{2k}c_mc_nc_pc_q$$

$$= \{6\delta_{rs}(\delta_{mn}\delta_{pq} + \delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}) - \delta_{rm}(\delta_{sn}\delta_{pq} + \delta_{sp}\delta_{nq} + \delta_{sq}\delta_{np})$$
$$- \delta_{sm}(\delta_{rn}\delta_{pq} + \delta_{rp}\delta_{nq} + \delta_{rq}\delta_{np}) - \delta_{rn}(\delta_{sp}\delta_{mq} + \delta_{sq}\delta_{mp}) - \delta_{rp}\delta_{sq}\delta_{mn}$$
$$- \delta_{sn}(\delta_{rp}\delta_{mq} + \delta_{rq}\delta_{mp}) - \delta_{sp}\delta_{rq}\delta_{mn}\}\frac{2^{k+3}(k+2)!}{105\sqrt{2\pi}}$$

As the Gaussians are still factorized in the new variables, the integrals reduce to combinations of integrals of the form

$$\left[\int \mathrm{d}\Gamma \exp(-\Gamma^2)\Gamma^p\Gamma_q\Gamma_r \cdots\right]\left[\int \mathrm{d}\gamma \exp(-\gamma^2/4)\; G_{mn}(\gamma)\; \gamma^q\gamma_r\gamma_s \cdots\right].$$

Apart from a trivial change of scale, these integrals are of the same type as (6.21) and (6.22).

In conclusion, all the integrals necessary for the calculation of the generalized frictions are of the form (6.21) and (6.22). They can be calculated very easily: the results are collected in table 6.1. The method by which they are obtained is explained in Appendix A2.

From here on, the calculation of the generalized frictions is a quite straightforward, though tedious matter. We illustrate the procedure for two simple, typical examples in Appendix A3. We now list and discuss the results of these calculations.

We first give the form of the coefficients in the ion–electron collision term, (2.8.18), (6.19), evaluated in the 21 M approximation,

$$L_n = -\frac{3}{\sqrt{10}} h_n^{\mathrm{e}(3)} + \frac{15}{2\sqrt{70}} h_n^{\mathrm{e}(5)},$$

$$M_{mn} = \delta_{mn} - \frac{6}{5\sqrt{2}} h_{mn}^{\mathrm{e}(2)} + \frac{3}{\sqrt{7}} h_{mn}^{\mathrm{e}(4)},$$

$$N_{kmn} = \frac{3}{\sqrt{10}}\left(\delta_{kp}\delta_{mn} + \delta_{kn}\delta_{mp} + \delta_{km}\delta_{np}\right)\left(h_p^{\mathrm{e}(3)} - \frac{\sqrt{7}}{2} h_p^{\mathrm{e}(5)}\right),$$

$$\hat{L}_{ns} = \delta_{ns} - \frac{\sqrt{2}}{5} h_{ns}^{\mathrm{e}(2)} + \frac{3}{5\sqrt{7}} h_{ns}^{\mathrm{e}(4)}. \tag{6.24}$$

We now list the collisional terms entering the non-hydrodynamical equations (5.12) and (5.22)–(5.27). They are collected in table 6.2; the numerical values of the coefficients of these formulae are given in table 6.3. It is essential to discuss and understand the structure of these formulae.

We first note that each generalized friction term $Q_{\cdots}^{\alpha(M)}$ consists of a *linear* part and a *non-linear* part. The former is defined by a matrix C_{MN}^{e} for the electron moments and by a matrix C_{MN}^{i} for the ion moments. The non-linear part is defined by an array of coefficients $D_{M\,|\,NP}^{\mathrm{e}}$, respectively $D_{M\,|\,NP}^{\mathrm{i}}$.

Table 6.2
The generalized frictions.

Vector moments

$$Q_r^{e(2n+1)} = \sum_{m=0}^{3} C^{e}_{2n+1,2m+1} h_r^{e(2m+1)} + \sum_{m=0}^{2}\sum_{p=1}^{2} D^{e}_{2n+1|2m+1,2p} h_s^{e(2m+1)} h_{sr}^{e(2p)},$$

Convention: $Q_r^{e(1)} \equiv Q_r^{(1)}$; $\quad h_r^{e(1)} \equiv h_r^{(1)}$ $\qquad n = 0, 1, 2, 3$

$$Q_r^{i(2n+1)} = \sum_{m=1}^{3} C^{i}_{2n+1,2m+1} h_r^{i(2m+1)} + \sum_{m=1}^{2}\sum_{p=1}^{2} D^{i}_{2n+1|2m+1,2p} h_s^{i(2m+1)} h_{sr}^{i(2p)},$$

$n = 1, 2, 3$

Tensor moments

$$Q_{rs}^{e(2n)} = \sum_{m=1}^{3} C^{e}_{2n,2m} h_{rs}^{e(2m)}$$

$$+ \mathscr{T}_{rs|jk}\left(\sum_{m=1}^{1}\sum_{p=0}^{1} D^{e}_{2n|2m+1,2p+1} h_j^{e(2m+1)} h_k^{e(2p+1)} + D^{e}_{2n|2,2} h_{jq}^{e(2)} h_{kq}^{e(2)} \right), \qquad n = 1, 2, 3$$

$$Q_{rs}^{i(2n)} = \sum_{m=1}^{3} C^{i}_{2n,2m} h_{rs}^{i(2m)} + \mathscr{T}_{rs|jk} D^{i}_{2n|2,2} h_{jq}^{i(2)} h_{kq}^{i(2)}, \qquad n = 1, 2, 3$$

$$\mathscr{T}_{rs|jk} = \tfrac{1}{2}\left(\delta_{rj}\delta_{sk} + \delta_{rk}\delta_{sj} - \tfrac{2}{3}\delta_{rs}\delta_{jk} \right)$$

Next, we note that each of these coefficients may have a contribution from the like-particle collisions and one from the unlike-particle collisions. We therefore write

$$C^{e}_{MN} = C^{ee}_{MN} + C^{ei}_{MN}, \qquad C^{i}_{MN} = C^{ii}_{MN} + C^{ie}_{MN},$$

$$D^{e}_{M|NP} = D^{ee}_{M|NP} + D^{ei}_{M|NP}, \; D^{i}_{M|NP} = D^{ii}_{M|NP} + D^{ie}_{M|NP}. \tag{6.25}$$

It turns out that the *ion–electron* (ie) *contributions to all these terms are of order* $\mu(= m_e/m_i)$, *hence they are negligible by our general standards*:

$$C^{ie}_{MN} \approx 0, \qquad D^{ie}_{M|NP} \approx 0. \tag{6.26}$$

It can be clearly seen from the structure of the kinetic equations, and from the two examples treated in Appendix A3, that the "special" moment $h_r^{(1)}$, i.e.

Table 6.3A
Coefficients of the linear electron collision matrix; $C^e_{MN} = -\tau_e^{-1} c^e_{MN} = C^e_{NM}$.

Vector moment couplings

$$c^e_{11} = 1$$

$$c^e_{13} = \frac{3}{\sqrt{10}} \qquad c^e_{33} = \frac{13 + 4\sqrt{2}\, Z^{-1}}{10}$$

$$c^e_{15} = -\frac{15}{2\sqrt{70}} \qquad c^e_{35} = -\frac{69 + 12\sqrt{2}\, Z^{-1}}{20\sqrt{7}} \qquad c^e_{55} = \frac{433\sqrt{2} + 360 Z^{-1}}{280\sqrt{2}}$$

$$c^e_{17} = \frac{\sqrt{105}}{12} \qquad c^e_{37} = \frac{33 + 3\sqrt{2}\, Z^{-1}}{4\sqrt{42}} \qquad c^e_{57} = -\frac{1077 + 421\sqrt{2}\, Z^{-1}}{280\sqrt{6}}$$

$$c^e_{77} = \frac{35484 + 45131\sqrt{2}\, Z^{-1}}{20160}$$

Tensor moment couplings

$$c^e_{22} = \frac{6 + 3\sqrt{2}\, Z^{-1}}{5}$$

$$c^e_{24} = -\frac{36 + 9\sqrt{2}\, Z^{-1}}{10\sqrt{14}} \qquad c^e_{44} = \frac{204\sqrt{2} + 205\, Z^{-1}}{140\sqrt{2}}$$

$$c^e_{26} = \frac{24 + 3\sqrt{2}\, Z^{-1}}{8\sqrt{14}} \qquad c^e_{46} = -\frac{2232\sqrt{2} + 1147\, Z^{-1}}{1680\sqrt{2}} \qquad c^e_{66} = \frac{6768\sqrt{2} + 10935\, Z^{-1}}{4032\sqrt{2}}$$

Table 6.3B
Coefficients of the linear ion collision matrix; $C^i_{MN} = -\tau_i^{-1} c^i_{MN} = C^i_{NM}$.

Vector moment couplings

$$c^i_{33} = \frac{2\sqrt{2}}{5}$$

$$c^i_{35} = -\frac{3\sqrt{14}}{35} \qquad c^i_{55} = \frac{9\sqrt{2}}{14}$$

$$c^i_{37} = \frac{3}{4\sqrt{21}} \qquad c^i_{57} = -\frac{421}{280\sqrt{3}} \qquad c^i_{77} = \frac{45131}{10080\sqrt{2}}$$

Tensor moment couplings

$$c^i_{22} = \frac{3\sqrt{2}}{5}$$

$$c^i_{24} = -\frac{9\sqrt{7}}{70} \qquad c^i_{44} = \frac{41\sqrt{2}}{56}$$

$$c^i_{26} = \frac{3\sqrt{7}}{56} \qquad c^i_{46} = -\frac{1147}{1680\sqrt{2}} \qquad c^i_{66} = \frac{10935}{4032\sqrt{2}}$$

Table 6.3C
Coefficients of the non-linear electron collision matrix; $D^{e}_{M|NP} = \tau_e^{-1} d^{e}_{M|NP}$.

Vector moment equations

$$d^{e}_{1|1,2} = \frac{3\sqrt{2}}{5}$$

$$d^{e}_{3|1,2} = \frac{1}{5\sqrt{5}} \qquad d^{e}_{3|3,2} = \tfrac{1}{5} Z^{-1}$$

Tensor moment equations

$$d^{e}_{2|1,1} = \frac{\sqrt{2}}{5} \qquad d^{e}_{2|2,2} = \tfrac{13}{35} Z^{-1}$$

$$d^{e}_{2|1,3} = -\frac{3\sqrt{5}}{50} \qquad d^{e}_{2|3,3} = \tfrac{45}{40} Z^{-1}$$

the dimensionless current, enters the generalized frictions only through the unlike-particle collisions. Hence, because of (6.26), this moment does not contribute to any term $Q^{i(M)}_{\dots}$. On the other hand, it is involved in all electron terms $Q^{e(M)}_{\dots}$ as well as in $Q^{(1)}_r$, on the same footing as all the other vectorial Hermitian moments. We therefore achieve a better symmetry and economy in the presentation by grouping together the current with the electron Hermitian moments, and making the notational convention

$$Q^{e(1)}_r = Q^{(1)}_r, \qquad h^{e(1)}_r = h^{(1)}_r \tag{6.27}$$

This convention will be used only when the system of variables (4.2) [rather than (4.3)] is adopted, and when there is no possible confusion with the electron flux (4.5).

We now discuss in some detail the *linear collision operator*: it has a number of remarkable symmetry properties. We note that *it couples only ion moments among themselves and electron moments (including $h^{(1)}_r$) among themselves*: this is expressed by the presence of two separate matrices C^{e}_{MN}, C^{i}_{MN}. Next, within each group, the linear collision operator *couples each moment to all other moments of the same tensorial nature*. This is expressed by the property

$$C^{e}_{2m,2n+1} = C^{e}_{2m+1,2n} = 0,$$

$$C^{i}_{2m,2n+1} = C^{i}_{2m+1,2n} = 0. \tag{6.28}$$

The collision matrix is symmetric:

$$C^{e}_{MN} = C^{e}_{NM}, \qquad C^{i}_{MN} = C^{i}_{NM}. \tag{6.29}$$

This property is quite important: it implies that the eigenvalues of the collision operator are all *real*. It turns out, moreover, that the eigenvalues are *definite negative*. This expresses the fact that the collisions bring any small deviation monotonically back to equilibrium.

Looking now at the detailed form of the coefficients, we note the following regularities. The electron collision coefficients are all of the form

$$C^{\mathrm{e}}_{MN} = -\frac{1}{\tau_{\mathrm{e}}}\left(\alpha_{MN} + Z^{-1}\beta_{MN}\right) \equiv -\frac{1}{\tau_{\mathrm{e}}}\, c^{\mathrm{e}}_{MN}. \tag{6.30}$$

The conventional minus sign in this definition turns out to be very convenient. Here α_{MN}, β_{MN}, c^{e}_{MN} are purely numerical coefficients: they are collected in table 6.3A. The first term, α_{MN}, is the contribution of the electron–ion collisions, whereas β_{MN} comes from the electron–electron collisions. The factor Z^{-1} is a signature of the latter. It follows that *in a plasma with highly charged ions* ($Z \gg 1$), *the electron–ion contribution dominates the electron–electron contribution*:

$$\left|C^{\mathrm{ei}}_{MN}\right| \gg \left|C^{\mathrm{ee}}_{MN}\right|, \qquad [Z \gg 1] \tag{6.31}$$

Clearly, the matrix elements c^{e}_{1N} have a vanishing electron–electron contribution (see eq. 3.4.13).

The ionic matrix elements (table 6.3B) contain only an ion–ion contribution. Hence, combining (6.3), (6.9) and (6.30), we find

$$C^{\mathrm{i}}_{MN} = -\frac{1}{\tau_{\mathrm{i}}}\, c^{\mathrm{i}}_{MN}, \tag{6.32}$$

with

$$c^{\mathrm{i}}_{MN} = \beta_{MN}. \tag{6.33}$$

For easy reference, we list here explicitly the *linear* parts of the *generalized frictions* involved in the 21M approximation:

$$\tau_{\mathrm{e}} Q^{(1)}_r = -c^{\mathrm{e}}_{11} h^{(1)}_r - c^{\mathrm{e}}_{13} h^{\mathrm{e}(3)}_r - c^{\mathrm{e}}_{15} h^{\mathrm{e}(5)}_r,$$

$$\tau_{\mathrm{e}} Q^{\mathrm{e}(3)}_r = -c^{\mathrm{e}}_{31} h^{(1)}_r - c^{\mathrm{e}}_{33} h^{\mathrm{e}(3)}_r - c^{\mathrm{e}}_{35} h^{\mathrm{e}(5)}_r,$$

$$\tau_{\mathrm{e}} Q^{\mathrm{e}(5)}_r = -c^{\mathrm{e}}_{51} h^{(1)}_r - c^{\mathrm{e}}_{53} h^{\mathrm{e}(3)}_r - c^{\mathrm{e}}_{55} h^{\mathrm{e}(5)}_r,$$

$$\tau_{\mathrm{i}} Q^{\mathrm{i}(3)}_r = -c^{\mathrm{i}}_{33} h^{\mathrm{i}(3)}_r - c^{\mathrm{i}}_{35} h^{\mathrm{i}(5)}_r,$$

$$\tau_i Q_r^{i(5)} = -c_{53}^i h_r^{i(3)} - c_{55}^i h_r^{i(5)},$$

$$\tau_e Q_{rs}^{e(2)} = -c_{22}^e h_{rs}^{e(2)} - c_{24}^e h_{rs}^{e(4)},$$

$$\tau_e Q_{rs}^{e(4)} = -c_{42}^e h_{rs}^{e(2)} - c_{44}^e h_{rs}^{e(4)},$$

$$\tau_i Q_{rs}^{i(2)} = -c_{22}^i h_{rs}^{i(2)} - c_{24}^i h_{rs}^{i(4)},$$

$$\tau_i Q_{rs}^{i(4)} = -c_{42}^i h_{rs}^{i(2)} - c_{44}^i h_{rs}^{i(4)}. \tag{6.34}$$

In forthcoming chapters we will also need the expressions of the generalized frictions entering the particle flux equations (5.16). These are related to $Q_r^{(1)}$ by eq. (5.18). We thus find

$$\tau_e Q_r^{e(1)} = c_{11}^e h_r^{(1)} + c_{13}^e h_r^{e(3)} + c_{15}^e h_r^{e(5)}$$

$$\tau_i Q_r^{i(1)} = \frac{1}{aA}\left(-c_{11}^e h_r^{(1)} - c_{13}^e h_r^{e(3)} - c_{15}^e h_r^{e(5)}\right) \tag{6.35}$$

The coefficients c_{MN}^e, c_{MN}^i are given in table 6.3 as functions of the charge number Z. The following symbols are used in (6.35):

$$a = \left(\frac{T_i}{m_i}\frac{m_e}{T_e}\right)^{1/2}, \tag{6.36}$$

$$A = \frac{1}{Z\mu}\frac{\tau_e}{\tau_i} = \frac{|\Omega_e|\tau_e}{\Omega_i \tau_i}. \tag{6.37}$$

The second form of the parameter A will prove to be quite useful later.

If we prefer to express the generalized frictions in terms of the particle fluxes, we simply replace the dimensionless electric current $h_r^{(1)}$ by

$$h_r^{(1)} = a h_r^{i(1)} - h_r^{e(1)}. \tag{6.38}$$

Let us stress a rather obvious point, which turns out to be very important in connection with the neoclassical transport theory developed in forthcoming chapters. *The relations* (6.34) *between generalized frictions and Hermitian moments (or "fluxes") are intrinsic, in the sense that they only involve the matrix elements of the collision operator* *.

* The reason we insist on this point is that Braginskii (1965) uses, in his classical work, a different representation of the friction term, relating it to the electric current and to the *temperature gradient* (rather than the heat flux). Such a relation involves the additional use of a transport equation and therefore depends on the collisionality regime (see below, section 5.7). This fact was also pointed out by Hirshman and Sigmar (1981).

We now briefly discuss the *non-linear contributions* to the collision matrix. We have selected among these only the coefficients entering the moment equations in the 13 M approximation. All these coefficients are again proportional to τ_α^{-1}, hence

$$D^{e}_{M|NP} = \frac{1}{\tau_e} d^{e}_{M|NP}. \tag{6.39}$$

We note here that the electron–ion and the electron–electron collisions give rise to different contributions:

$$d^{e}_{2n+1|1,2m} = d^{ei}_{2n+1|1,2m}, \qquad d^{e}_{2n+1|3,2m} = d^{ee}_{2n+1|3,2m},$$

$$d^{e}_{2n|1,2m+1} = d^{ei}_{2n|1,2m+1}, \qquad d^{e}_{2n|m,m} = d^{ee}_{2n|m,m}. \tag{6.40}$$

The electron–electron contributions are again recognizable by their signature Z^{-1}.

Finally, the non-linear contributions to the ion moment equations originate only from the ion–ion collisions.

Before closing this chapter, we discuss an important collisional contribution which is not listed in table 6.2: this is the *collisional heat exchange* $Q^{(2)}$ appearing in the temperature equations (5.3), (5.4). By using the same methods as before we find, using (5.5), (6.17)–(6.19), (6.24),

$$Q^{(2)} = -\frac{2Z}{\tau_e}\mu(T_i - T_e). \tag{6.41}$$

We discarded here higher-order terms, which are all multiplied by μ. We may also use (5.6) and (6.12) to check directly relation (5.8).

This result shows that the unlike-particle collisions tend to equalize the ion and electron temperatures. However, the process is *extremely slow*, as the corresponding relaxation time is proportional to (τ_e/μ). This result provides the final justification of the concept of a *local plasma equilibrium*, characterized by two different temperatures, considered as quasi-conserved quantities.

Appendix 4A.1. Derivation of the moment equations

As a typical example of the calculations involved in the derivation of the moment equations, we consider here the equation for the dimensionless heat flux, $h_r^{\alpha(3)}$.

The tricky point in the calculation is that one should not switch over too early to the dimensionless variable $\boldsymbol{c}$. We thus write, by using the definition in table **G**1.4.1,

$$\partial_t h_r^{\alpha(3)} = \partial_t \left\{ n_\alpha^{-1} \int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{v})\ H_r^{(3)}\left[(m_\alpha/T_\alpha)^{1/2} (\boldsymbol{v} - \boldsymbol{u}^\alpha) \right] \right\}$$

$$= \partial_t \left[\frac{1}{\sqrt{10}\, n_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} \int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{v})\ (v_r - u_r^\alpha) \right.$$

$$\left. \times \left[|\boldsymbol{v} - \boldsymbol{u}^\alpha|^2 - 5(T_\alpha/m_\alpha) \right] \right]. \tag{A1.1}$$

We now remember that on the right-hand side, not only f^α, but also the plasmadynamical variables n_α, $\boldsymbol{u}^\alpha$, T_α depend on time, hence, using the kinetic equation (**3**.3.1), we have

$$\partial_t h_r^{\alpha(3)} = \left(n_\alpha T_\alpha^{3/2} \right) h_r^{\alpha(3)}\ \partial_t \left(n_\alpha T_\alpha^{3/2} \right)^{-1}$$

$$- (\partial_t u_r^\alpha) \frac{1}{\sqrt{10}\, n_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{3/2} \int \mathrm{d}\boldsymbol{v} f^\alpha \left(|\boldsymbol{v} - \boldsymbol{u}^\alpha|^2 - 5(T_\alpha/m_\alpha) \right)$$

$$- (\partial_t u_n^\alpha) \frac{2}{\sqrt{10}\, n_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{3/2} \int \mathrm{d}\boldsymbol{v} f^\alpha\ (v_r - u_r^\alpha)(v_n - u_n^\alpha)$$

$$- (\partial_t T_\alpha) \frac{5}{\sqrt{10}\, m_\alpha n_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{3/2} \int \mathrm{d}\boldsymbol{v} f^\alpha\ (v_r - u_r^\alpha)$$

$$+ \frac{1}{\sqrt{10}\, n_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{3/2} \int \mathrm{d}\boldsymbol{v}\ (v_r - u_r^\alpha) \left(|\boldsymbol{v} - \boldsymbol{u}^\alpha|^2 - 5(T_\alpha/m_\alpha) \right)$$

$$\times \left\{ -v_m \nabla_m f^\alpha - \frac{e_\alpha}{m_\alpha} \left[\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \wedge \boldsymbol{B} \right] \cdot \partial f^\alpha + \mathscr{K}^\alpha \right\}. \tag{A1.2}$$

The time derivatives in the four first terms are evaluated by using the plasmadynamical equations (**3**.4.5), (**3**.4.10), (**3**.4.11). In the last term, we treat

separately the contributions of the various terms of the kinetic equation. We consider, in particular,

$$\int \mathrm{d}\boldsymbol{v}\ (v_r - u_r^\alpha)\big(|\boldsymbol{v}-\boldsymbol{u}^\alpha|^2 - 5(T_\alpha/m_\alpha)\big)v_m \nabla_m f^\alpha$$

$$= \nabla_m \int \mathrm{d}\boldsymbol{v}\ (v_r - u_r^\alpha)\big(|\boldsymbol{v}-\boldsymbol{u}^\alpha|^2 - 5(T_\alpha/m_\alpha)\big)\big[(v_m - u_m^\alpha) + u_m^\alpha\big] f^\alpha$$

$$+ (\nabla_m u_r^\alpha)\int \mathrm{d}\boldsymbol{v}\ \big(|\boldsymbol{v}-\boldsymbol{u}^\alpha|^2 - 5(T_\alpha/m_\alpha)\big)\big[(v_m - u_m^\alpha) + u_m^\alpha\big] f^\alpha$$

$$+ 2(\nabla_m u_n^\alpha)\int \mathrm{d}\boldsymbol{v}\ (v_r - u_r^\alpha)(v_n - u_n^\alpha)\big[(v_m - u_m^\alpha) + u_m^\alpha\big] f^\alpha$$

$$+ (5/m_\alpha)(\nabla_m T_\alpha)\int \mathrm{d}\boldsymbol{v}\ (v_r - u_r^\alpha)\big[(v_m - u_m^\alpha) + u_m^\alpha\big] f^\alpha. \qquad \text{(A1.3)}$$

At this point it is safe to go over to the dimensionless variables c_r. As an example, the first term on the right-hand side is treated as

$$\nabla_m \int \mathrm{d}\boldsymbol{v}\ (v_r - u_r^\alpha)\big(|\boldsymbol{v}-\boldsymbol{u}^\alpha|^2 - 5(T_\alpha/m_\alpha)\big)\big[(v_m - u_m^\alpha) + u_m^\alpha\big] f^\alpha$$

$$= \nabla_m \Big\{ n_\alpha (T_\alpha/m_\alpha)^2 \Big[\int \mathrm{d}\boldsymbol{c}\ c_r c_m (c^2 - 5)\ \phi^0 (1 + \chi^\alpha)$$

$$+ u_m^\alpha \int \mathrm{d}\boldsymbol{c}\ c_r (c^2 - 5)\ \phi^0 (1 + \chi^\alpha) \Big] \Big\}$$

$$= \nabla_m \big[n_\alpha (T_\alpha/m_\alpha)^2 \big(2\sqrt{7}\, h_{rm}^{\alpha(4)} + 2\sqrt{2}\, h_{rm}^{\alpha(2)} + \sqrt{10}\, u_m^\alpha h_r^{\alpha(3)} \big) \big]. \qquad \text{(A1.4)}$$

The last step is obtained by expressing the polynomials in the integrands in terms of Hermite polynomials,

$$c_r c_m (c^2 - 5) = 2\sqrt{7}\, H_{rm}^{(4)}(\boldsymbol{c}) + 2\sqrt{2}\, H_{rm}^{(2)}(\boldsymbol{c}),$$

$$c_r (c^2 - 5) = \sqrt{10}\, H_r^{(3)}(\boldsymbol{c}).$$

Let us make a special note about the contribution of the third term on the rhs of (A1.3):

$$\int \mathrm{d}\boldsymbol{c}\ \phi^0 (1 + \chi^\alpha) c_r c_m c_n = \sqrt{\tfrac{2}{5}} \big(h_r^{\alpha(3)} \delta_{mn} + h_m^{\alpha(3)} \delta_{rn} + h_n^{\alpha(3)} \delta_{rm} \big). \qquad \text{(A1.5)}$$

This result is a consequence of the *truncation* (3.19): it is valid in all three approximations (13 M, 21 M, 29 M) considered here. If higher anisotropies were retained, we would also have a term proportional to $h^{\alpha(3)}_{rmn}$.

We believe that these indications are sufficient for the understanding of the derivation of the moment equations.

Appendix 4A.2. Proof of the results of table 6.1.

All the integration formulae of table 6.1 can be easily obtained by using arguments of tensorial symmetry. They will be illustrated by two typical examples. Consider first

$$J^{(k)}_{mnpq} = (2\pi)^{-3/2} \int \mathrm{d}\Gamma\, \mathrm{e}^{-\Gamma^2}\, \Gamma^{2k} \Gamma_m \Gamma_n \Gamma_p \Gamma_q .$$

Clearly, this object is a tensor of rank 4, completely symmetric in its indices. After integration, there is no privileged vector or tensor available; therefore, the tensor $J^{(k)}_{mnpq}$ can only be constructed by combinations of the unit tensor. We conclude from this discussion that

$$J^{(k)}_{mnpq} = B^{(k)} \left(\delta_{mn}\delta_{pq} + \delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np} \right),$$

where $B^{(k)}$ is a purely numerical scalar. In order to calculate its value, we perform twice the trace of both sides:

$$J^{(k)}_{mnrr} = 5B^{(k)}\delta_{mn}, \qquad J^{(k)}_{rrss} = 3 \cdot 5B^{(k)}.$$

The left-hand side is easily calculated, by using standard Gaussian integrals,

$$J^{(k)}_{rrss} = (2\pi)^{-3/2} \int \mathrm{d}\Gamma\, \mathrm{e}^{-\Gamma^2}\, \Gamma^{2k+4}$$

$$= 4\pi (2\pi)^{-3/2} \int_0^\infty \mathrm{d}\Gamma\, \mathrm{e}^{-\Gamma^2}\, \Gamma^{2k+6} = \frac{3 \cdot 5 \cdot 7 \cdots (2k+5)}{2^{k+(7/2)}},$$

from which we find the value of $B^{(k)}$ given in table 6.1.

The integrals involving the Landau tensor are somewhat more complicated. Consider, for instance,

$$L^{(k)}_{rs|mn} = (2\pi)^{-3/2} \int \mathrm{d}\boldsymbol{c}\, \mathrm{e}^{-c^2/2}\, G_{rs}(\boldsymbol{c})\, c^{2k} c_m c_n .$$

This is a tensor of rank 4, symmetric in the group (rs) and in (mn), but not in any cross-permutation between these groups. It must also be constructed by combinations of unit tensors. It must therefore be of the form

$$L^{(k)}_{rs\,|\,mn} = B^{(k)}_1 \delta_{rs}\delta_{mn} + B^{(k)}_2 (\delta_{rm}\delta_{sn} + \delta_{rn}\delta_{sm}).$$

Next, we note that (2.7.8) implies

$$L^{(k)}_{rp\,|\,pn} = L^{(k)}_{rp\,|\,mp} = 0.$$

In order to determine the coefficients, we first take the trace over (rs) and (mn):

$$L^{(k)}_{pp\,|\,qq} = 3\big(3B^{(k)}_1 + 2B^{(k)}_2\big).$$

Next, we calculate

$$L^{(k)}_{pq\,|\,pq} = 0 = B^{(k)}_1 + 4B^{(k)}_2.$$

We thus find

$$B^{(k)}_1 = -4B^{(k)}_2$$

and we easily calculate

$$L^{(k)}_{pp\,|\,qq} = (2\pi)^{-3/2} \int \mathrm{d}c\ \mathrm{e}^{-c^2/2} \frac{c^2\delta_{pp} - c_p c_p}{c^3} c^{2k} c_q c_q$$

$$= 4\pi(2\pi)^{-3/2} \int_0^\infty \mathrm{d}c\ \mathrm{e}^{-c^2/2}\ 2c^{2k+3}$$

$$= \frac{2^{k+3}(k+1)!}{15\sqrt{2\pi}},$$

from which we determine

$$B^{(k)}_2 = -\frac{2^{k+2}(k+1)!}{15\sqrt{2\pi}}$$

and thus the result given in table 6.1.

Appendix 4A.3. Collisional contributions to the moment equations

We illustrate the calculations involved in the evaluation of these contributions by considering two simple examples.

(a) The friction force $Q_r^{(1)}$. From eqs. (3.4.9), (3.4.13), (5.7) and (6.12)–(6.14) we obtain, with $\psi(\boldsymbol{c}) = c_r$,

$$\mathbb{Q}_r^{\mathrm{ei}(1)} = \int \mathrm{d}\boldsymbol{c}\; G_{jk}(\boldsymbol{c})\; \mathscr{H}_{jk}^{\mathrm{e}}(\boldsymbol{c})$$

$$= \int \mathrm{d}\boldsymbol{c}\; G_{rk}(\boldsymbol{c}) \left(\frac{\partial \phi^{\mathrm{e}}(\boldsymbol{c})}{\partial c_k} + h_n^{(1)}\, \frac{\partial^2 \phi^{\mathrm{e}}(\boldsymbol{c})}{\partial c_n\, \partial c_k} + \cdots \right). \tag{A3.1}$$

For the present illustration, we only keep two terms in the Lorentz expansion (6.14) and evaluate $\phi^{\mathrm{e}}(\boldsymbol{c})$ in the 13 M approximation (3.20). Using table **G1**.4.1, we get

$$\frac{\partial \phi^{\mathrm{e}}}{\partial c_i} = \phi^0 \left(\frac{1}{\sqrt{10}} h_k^{\mathrm{e}(3)}(c^2 - 5) + \frac{2}{\sqrt{2}} h_{kp}^{\mathrm{e}(2)} c_p + c_k[\ldots] \right)$$

$$\frac{\partial^2 \phi^{\mathrm{e}}}{\partial c_n\, \partial c_k} = \phi^0 \left(-\delta_{nk} - \frac{1}{\sqrt{10}} (c^2 - 7) \left(h_k^{\mathrm{e}(3)} c_n + \delta_{nk} h_p^{\mathrm{e}(3)} c_p \right) \right.$$

$$\left. - \frac{1}{\sqrt{2}} \left(2 h_{kp}^{\mathrm{e}(2)} c_p c_n + \delta_{nk} h_{pq}^{\mathrm{e}(2)} c_p c_q - 2 h_{nk}^{\mathrm{e}(2)} \right) + c_k[\ldots] \right).$$

The terms proportional to c_k were not written down, because they do not contribute to the integral (see eq. **2**.7.8). Substituting these terms into (A3.1) (and writing out only terms of even parity), we find

$$\mathbb{Q}_r^{\mathrm{ei}(1)} = \int \mathrm{d}\boldsymbol{c}\; G_{rk}(\boldsymbol{c}) \left[\frac{1}{\sqrt{10}} h_k^{\mathrm{e}(3)}(c^2 - 5) \right.$$

$$- h_n^{(1)} \left(\delta_{nk} + \frac{1}{\sqrt{2}} \left(2 h_{kp}^{\mathrm{e}(2)} c_p c_n + \delta_{nk} h_{pq}^{\mathrm{e}(2)} c_p c_q \right. \right.$$

$$\left. \left. - 2 h_{nk}^{\mathrm{e}(2)} \right) \right) \left. \right] \phi^0(c).$$

All these integrals are of the type listed in table 6.1; we thus find

$$\mathbb{Q}_r^{ei(1)} = \frac{4}{3\sqrt{2\pi}}\left(-\frac{3}{\sqrt{10}}h_r^{e(3)} - h_r^{(1)} + \frac{6}{5\sqrt{2}}h_{rn}^{e(2)}h_n^{(1)}\right). \tag{A3.2}$$

(b) The electron–electron collision term $Q_{rs}^{ee(2)}$. As an illustration of the treatment of a like-particle collision term, we consider eq. (6.6) with $\psi(\boldsymbol{c}_1) = H_{rs}(\boldsymbol{c}_1)$; hence (see eq. 6.23),

$$\frac{\partial\psi}{\partial c_{1m}} = \frac{1}{\sqrt{2}}\left(c_{1r}\delta_{ms} + c_{1s}\delta_{mr} - \tfrac{2}{3}\delta_{rs}c_{1m}\right)$$

$$= \frac{1}{\sqrt{2}}\left[\left(\Gamma_r + \tfrac{1}{2}\gamma_r\right)\delta_{ms} - \tfrac{1}{3}\delta_{rs}\left(\Gamma_m + \tfrac{1}{2}\gamma_m\right)\right] + [r \leftrightarrow s].$$

In the product $\phi^e\phi^e$ we retain (in this illustration) only terms linear in the moment $h_{rs}^{e(2)}$:

$$(\partial_{1n} - \partial_{2n})\phi^0(c_1)\ \phi^0(c_2)\left(1 + \frac{1}{\sqrt{2}}h_{pq}^{e(2)}H_{pq}^{(2)}(\boldsymbol{c}_1)\right)\left(1 + \frac{1}{\sqrt{2}}h_{uv}^{e(2)}H_{uv}^{(2)}(\boldsymbol{c})\right)$$

$$\approx \phi^0(c_1)\ \phi^0(c_2)\left[\frac{1}{\sqrt{2}}h_{pq}^{e(2)}\left(c_{1p}\delta_{nq} + c_{1q}\delta_{np} - c_{2q}\delta_{np} - c_{2p}\delta_{nq}\right)\right.$$

$$\left. + \gamma_n[\dots]\right]$$

$$= (2\pi)^{-3}\,e^{-\Gamma^2}\,e^{-\gamma^2/4}\left(\frac{1}{\sqrt{2}}h_{pq}^{e(2)}\left(\gamma_p\delta_{nq} + \gamma_q\delta_{np}\right) + \cdots\right).$$

Substituting these results into (6.5), the latter reduces to

$$\mathbb{Q}_{rs}^{ee(2)} = \frac{1}{\sqrt{2}\sqrt{2}\,(2\pi)^3}\int d\Gamma\, e^{-\Gamma^2}\int d\gamma\, e^{-\gamma^2/4}$$

$$\times\left(\tfrac{1}{2}\gamma_r\delta_{ms} + \tfrac{1}{2}\gamma_s\delta_{mr}\right)G_{mn}(\gamma)\ h_{pq}^{e(2)}\left(\gamma_p\delta_{nq} + \gamma_q\delta_{np}\right).$$

These integrals are again of the standard type of table 6.1 and we obtain

$$\mathbb{Q}_{rs}^{ee(2)} = \frac{2}{15\sqrt{\pi}}\cdot 6h_{rs}^{e(2)}. \tag{A3.3}$$

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5

The classical transport theory

5.1. *The linear transport regime*

Having derived a complete set of moment equations (within a well-defined approximation), we must now solve these equations. We thus derive expressions of the privileged moments in terms of the hydrodynamical variables. It is clear, however, that even this truncated problem, which is seriously reduced as compared to the original kinetic equation, is still formidable: we are faced with a set of coupled non-linear differential equations. We must therefore introduce additional simplifications before being able to solve these equations analytically.

We recall that the privileged non-hydrodynamical moments are driven directly by the hydrodynamical and electrodynamical "forces": these are the *gradients of the hydrodynamical variables* and the *Lorentz force* (see the discussion at the end of section **4**.4). In a first stage, we shall assume that these *driving forces are "weak"*.

In order to make this statement precise, we use the following argument. As all the moments $h^{\alpha(n)}_{\cdots}$ are dimensionless, all the terms in eqs. (**4**.5.12), (**4**.5.16), (**4**.5.22)–(**4**.5.27) have the dimensions of an inverse time. It follows that the source terms in those equations introduce the following characteristic times:

$$\tau_{T\alpha}^{-1} \equiv \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \frac{1}{T_\alpha} |\nabla T_\alpha|, \qquad \tau_\rho^{-1} \equiv \left(\frac{T_e}{m_e}\right)^{1/2} \frac{1}{\rho} |\nabla \rho|,$$

$$\tau_u^{-1} \equiv |\nabla \boldsymbol{u}|, \qquad \tau_E^{-1} \equiv \left(\frac{m_e}{T_e}\right)^{1/2} \frac{e}{m_e} E. \tag{1.1}$$

Let τ_H be the lower bound of all these times: τ_H will be called the *hydrodynamical time*. In other words, none of the times introduced in (1.1) may be shorter than τ_H *,

$$\tau_{T\alpha} \geqslant \tau_H, \qquad \tau_\rho \geqslant \tau_H, \qquad \tau_u \geqslant \tau_H, \qquad \tau_E \geqslant \tau_H.$$

* Nothing precludes one or more of these times from being infinite. This occurs whenever ρ, $\boldsymbol{u}$ or T_α happen to be spatially homogeneous, or in absence of an electromagnetic field.

We now introduce the main assumption, namely that *the hydrodynamical time is much longer than the collisional relaxation times* τ_e, τ_i, defined in (4.6.4) and (4.6.10):

$$\frac{\tau_e}{\tau_H} \ll 1, \qquad \frac{\tau_i}{\tau_H} \ll 1. \tag{1.2}$$

In an isothermal plasma, the second relation follows from the first. But in some cases (for instance, when the electron temperature is much larger than the ion temperature), τ_i may be of the same order, or smaller than τ_e *.

We denote by λ_H the greatest of the two ratios of (1.2),

$$\lambda_H = \mathrm{Max}\left(\frac{\tau_e}{\tau_H}, \frac{\tau_i}{\tau_H}\right) \ll 1. \tag{1.3}$$

The disparity (1.2) between the time scales (which is consistent with our qualitative discussion in section 4.1) is characteristic of the so-called *hydrodynamical regime* of evolution of the macroscopic quantities.

Next, we note that the hydrodynamical quantities, whose rate of change is determined by (4.5.1)–(4.5.4), evolve entirely on the hydrodynamical time scale, i.e. slowly. Indeed, eqs. (4.5.1) and (4.5.2) do not contain any collisional contributions, hence τ_α does not enter at all these equations. Equations (4.5.3) and (4.5.4) contain the collisional contribution $Q^{(2)}$ which, as shown in (4.6.41), is exceptionally small, and a non-linear term which is also small (see below). Hence, the hydrodynamic variables ρ, $\boldsymbol{u}$, T_α change very little in a time of order τ_α. These arguments reinforce our discussion in sections 4.1 and 4.2.

We now turn to the non-hydrodynamical moments. Their equations of evolution (4.5.12), (4.5.16) and (4.5.22)–(4.5.27) contain both slow and fast terms, i.e. terms involving gradients and electromagnetic forces, and terms coming from the collisions. Multiplying all the terms of these equations by τ_α and using the results of table 4.6.2, we see that the collisional contributions can be balanced against the hydrodynamical source terms if we assume that *the dimensionless moments* $h^{\alpha(p)}_{\dots}$ *are at most of order* λ_H. This remark allows us to start a perturbation theory, considering λ_H as a *small parameter*. We thus expand,

$$h^{\alpha(p)}_{\dots} = \sum_{r \geq 1} \lambda_H^r \, h^{\alpha(p)}_{[r]\dots}. \tag{1.4}$$

Recalling eqs. (4.3.1), (4.3.8) and (4.3.11), we note that the moments $h^{\alpha(p}_{\dots}$ are simply the coefficients of the Hermite representation of the deviation

* As an example, for hydrogen ions, $\tau_i < \tau_e$ whenever $(T_e/T_i) > \mu^{-1/3} \approx 12.24$.

$\chi^\alpha(c;\ x,\ t)$ from the local equilibrium state. Hence, (1.4) is equivalent to *an expansion of the distribution function in powers of the small parameter* λ_H:

$$f^\alpha(v;\ x,\ t) = f^{\alpha 0}(v;\ x,\ t)\Big(1 + \sum_q \lambda_H^q\, \chi^\alpha_{[q]}(v;\ x,\ t)\Big), \tag{1.5}$$

where $f^{\alpha 0}$ is the local plasma equilibrium distribution defined by (4.3.5).

Let us stress once more an essential point. In expansion (1.5), the exact plasmadynamical moments n_α, u^α, T_α enter as coefficients of the zeroth order reference function. Because of constraints (4.3.9), the presence of a deviation $\chi^\alpha_{[q]}$ adds no correction to the values of the conserved quantities as calculated in the local equilibrium. In other words, there is no expansion such as (1.4) for the hydrodynamical quantities.

The *linear transport theory* is concerned with the lowest order term in this expansion, i.e. $q = 1$. It therefore starts from the *moment equations, linearized in* λ_H. Before performing this linearization, we must devote special attention to the magnetic terms, which were not yet discussed. These terms introduce their own characteristic time scale, related to the *Larmor (or cyclotron) frequency* of the particles of species α, defined in (2.6.16). Explicitly, we have

$$\Omega_e = -\frac{eB}{m_e c}, \qquad \Omega_i = \frac{ZeB}{m_i c} = -Z\mu\Omega_e. \tag{1.6}$$

We emphasize again the fact that $\Omega_e < 0$ and $\Omega_i > 0$. Here we have a rather wide range of controllable variations. Compared to τ_α, the characteristic times $|\Omega_\alpha|^{-1}$ may be long or short. All these cases can be realized in present-day experiments. We therefore make no special assumptions at this stage concerning the order of magnitude of the characteristic product $\Omega_\alpha\tau_\alpha$: it will be treated here as a finite (zeroth order) quantity.

On the basis of this discussion, we linearize the moment equations (4.5.12), (4.5.22), (4.5.23), (4.5.25), (4.5.26), by discarding all terms other than the source terms, the magnetic terms and the linear collision terms. We also introduce the explicit expressions (4.6.34) of the collisional generalized friction forces. The resulting equations, *in the 21M approximation*, are then (we now set $\lambda_H = 1$ and revert to the simple notation $h^{\alpha(p)}_{[1]\cdots} \equiv h^{\alpha(p)}_{\cdots}$)

(A) Electron vector moments

$$\partial_t h_r^{(1)} - \Omega_e \varepsilon_{rmn} h_m^{(1)} b_n = -\frac{1}{\tau_e}\left(c^e_{11} h_r^{(1)} + c^e_{13} h_r^{e(3)} + c^e_{15} h_r^{e(5)}\right) + \frac{1}{\tau_e} g_r^{(1)},$$

$$\partial_t h_r^{e(2)} - \Omega_e \varepsilon_{rmn} h_m^{e(3)} b_n = -\frac{1}{\tau_e}\left(c^e_{31} h_r^{(1)} + c^e_{33} h_r^{e(3)} + c^e_{35} h_r^{e(5)}\right) + \frac{1}{\tau_e} g_r^{e(3)},$$

$$\partial_t h_r^{e(5)} - \Omega_e \varepsilon_{rmn} h_m^{e(5)} b_n = -\frac{1}{\tau_e}\left(c^e_{51} h_r^{(1)} + c^e_{53} h_r^{e(3)} + c^e_{55} h_r^{e(5)}\right). \tag{1.7}$$

(B) Electron tensor moments

$$\partial_t h^{e(2)}_{rs} - \Omega_e\left(\varepsilon_{rmn}h^{e(2)}_{sm} + \varepsilon_{smn}h^{e(2)}_{rm}\right)b_n$$

$$= -\frac{1}{\tau_e}\left(c^e_{22}h^{e(2)}_{rs} + c^e_{24}h^{e(4)}_{rs}\right) + \frac{1}{\tau_e}g^{e(2)}_{rs},$$

$$\partial_t h^{e(4)}_{rs} - \Omega_e\left(\varepsilon_{rmn}h^{e(4)}_{sm} + \varepsilon_{smn}h^{e(4)}_{rm}\right)b_n = -\frac{1}{\tau_e}\left(c^e_{42}h^{e(2)}_{rs} + c^e_{44}h^{e(4)}_{rs}\right). \tag{1.8}$$

(C) Ion vector moments

$$\partial_t h^{i(3)}_r - \Omega_i\varepsilon_{rmn}h^{i(3)}_m b_n = -\frac{1}{\tau_i}\left(c^i_{33}h^{i(3)}_r + c^i_{35}h^{i(5)}_r\right) + \frac{1}{\tau_i}g^{i(3)}_r,$$

$$\partial_t h^{i(5)}_r - \Omega_i\varepsilon_{rmn}h^{i(5)}_m b_n = -\frac{1}{\tau_i}\left(c^i_{53}h^{i(3)}_r + c^i_{55}h^{i(5)}_r\right). \tag{1.9}$$

(D) Ion tensor moments

$$\partial_t h^{i(2)}_{rs} - \Omega_i\left(\varepsilon_{rmn}h^{i(2)}_{sm} + \varepsilon_{smn}h^{i(2)}_{rm}\right)b_n$$

$$= -\frac{1}{\tau_i}\left(c^i_{22}h^{i(2)}_{rs} + c^i_{24}h^{i(4)}_{rs}\right) + \frac{1}{\tau_i}g^{i(2)}_{rs},$$

$$\partial_t h^{i(4)}_{rs} - \Omega_i\left(\varepsilon_{rmn}h^{i(4)}_{sm} + \varepsilon_{smn}h^{i(4)}_{rm}\right)b_n = -\frac{1}{\tau_i}\left(c^i_{42}h^{i(2)}_{rs} + c^i_{44}h^{i(4)}_{rs}\right). \tag{1.10}$$

The vector b_n is a unit vector along the magnetic field (as usual),

$$b_n = B_n/B. \tag{1.11}$$

The *dimensionless source terms* are defined as (see eqs. **4**.5.12, **4**.5.22, **4**.5.25)

$$g^{(1)}_r = \tau_e\left(\frac{m_e}{T_e}\right)^{1/2}\left(\frac{e}{m_e}E_r - \Omega_e\varepsilon_{rmn}u_m b_n + \frac{1}{m_e n_e}\nabla_r(n_e T_e)\right), \tag{1.12}$$

$$g^{e(3)}_r = -\sqrt{\tfrac{5}{2}}\,\tau_e\left(\frac{T_e}{m_e}\right)^{1/2}\frac{1}{T_e}\nabla_r T_e, \tag{1.13}$$

$$g^{e(2)}_{rs} = -\sqrt{2}\,\tau_e\,\mathscr{T}_{rs|pq}\;\nabla_p u_q, \tag{1.14}$$

$$g_r^{i(3)} = -\sqrt{\tfrac{5}{2}}\,\tau_i \left(\frac{T_i}{m_i}\right)^{1/2} \frac{1}{T_i} \nabla_r T_i, \tag{1.15}$$

$$g_{rs}^{i(2)} = -\sqrt{2}\,\tau_i\, \mathscr{T}_{rs|pq}\ \nabla_p u_q. \tag{1.16}$$

A natural factor τ_e or τ_i has been included in these definitions. These sources are therefore dimensionless quantities, proportional to (τ_α/τ_H).

We emphasize again the important fact that *the source terms of the non-privileged moment equations are identically zero*,

$$g_r^{e(5)} = g_r^{i(5)} = 0, \qquad g_{rs}^{e(4)} = g_{rs}^{i(4)} = 0. \tag{1.17}$$

The coefficients of the collision matrix c_{MN}^e, c_{MN}^i are listed in table **4**.6.3.

The set of moment equations (1.7)–(1.10) provides a complete description of the collisional plasma in the 21M approximation. According to the discussion in section **4**.4, it corresponds to the choice (**4**.4.2) for the one-fluid description. It was mentioned there that in some problems (such as those to be met in magnetically confined plasmas) the electric current may not be an appropriate variable, and one then prefers the choice (**4**.4.3). For completeness, we give here the corresponding moment equation for the *particle flux* $h_r^{\alpha(1)}$, which replaces the first equation (1.7) in that picture. From (**4**.5.16) we obtain

$$\partial_t h_r^{\alpha(1)} - \Omega_\alpha \varepsilon_{rmn} h_m^{\alpha(1)} b_n = Q_r^{\alpha(1)} + \frac{1}{\tau_\alpha} g_r^{\alpha(1)}, \tag{1.18}$$

or, more explicitly, for the case $\alpha = e$ (see eq. **4**.5.18):

$$\partial_t h_r^{e(1)} - \Omega_e \varepsilon_{rmn} h_m^{e(1)} b_n = \frac{1}{\tau_e}\left(c_{11}^e h_r^{(1)} + c_{13}^e h_r^{e(3)} + c_{15}^e h_r^{e(5)}\right) + \frac{1}{\tau_e} g_r^{e(1)}, \tag{1.19}$$

with the source term

$$g_r^{\alpha(1)} = -\tau_\alpha \left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \left(\frac{1}{m_\alpha n_\alpha} \nabla_r (n_\alpha T_\alpha) - \frac{e_\alpha}{m_\alpha} E_r\right). \tag{1.20}$$

The electric current still appears on the right-hand side of (1.19). If necessary, it can be eliminated by relation (**4**.4.6).

Equations (1.18)–(1.20) will be used later on, but *not* in the present chapter, where no more mention will be made of the particle flux $h_r^{\alpha(1)}$.

Before solving the moment equations, it is very important to have a good insight into their structure.

We first note that the linearized equations contain no term involving the gradients of the unknown moments. They are thus a set of *first-order ordinary differential equations*, which require only an initial condition, i.e. the value of the moments at $t = 0$.

Next, we note that the set of linearized equations splits up into *four groups of mutually uncoupled equations*, viz. the electron and ion vector equations and the electron and ion tensor equations. This feature represents an enormous simplification. Instead of dealing with a set of 39 equations, we only have to deal with small groups of equations which are easily solved.

Inside each group, we note a peculiar structure, which further simplifies the problem.

(a) Vector moment equations. Let $h_r^{(p)}$ $[r = x, y, z]$ be the three components of a specific vector moment. Consider first the case where there is no magnetic field, $\Omega_\alpha = 0$. In this situation, we see that there is no coupling between the components of the vectors:

$$\partial_t h_r^{(p)} = -\frac{1}{\tau}\sum_q c_{pq} h_r^{(q)} + \frac{1}{\tau} g_r^{(p)}, \quad r = x, y, z. \tag{1.21}$$

Hence the 3 equations (for $r = x, y, z$) are identical, except for the value of the source terms.

Consider now the case when there is a (constant) magnetic field, and choose a reference frame whose z-axis points along $\boldsymbol{B}$: $b_n = \delta_{nz}$. In this situation, we see from (1.7) and (1.9) that there still exists a partial decoupling. The equation for the *parallel component*, $h_z^{(p)}$, is unaffected by the magnetic field and thus obeys an equation identical to (1.21). The x- and y-components (*transverse components*) are mutually coupled by the magnetic term, and obey a set of 2 equations:

$$\begin{aligned} \partial_t h_x^{(p)} - \Omega h_y^{(p)} &= -\frac{1}{\tau}\sum_q c_{pq} h_x^{(q)} + \frac{1}{\tau} g_x^{(p)}, \\ \partial_t h_y^{(p)} + \Omega h_x^{(p)} &= -\frac{1}{\tau}\sum_q c_{pq} h_y^{(q)} + \frac{1}{\tau} g_y^{(p)}. \end{aligned} \tag{1.22}$$

(b) Tensor moment equations. Let $h_{rs}^{(p)}$ $(r, s = x, y, z)$ be any specified tensor moment. If $\Omega = 0$, the various components of the tensor are uncoupled and obey 6 separate equations that are identical, except for the sources:

$$\partial_t h_{rs}^{(p)} = -\frac{1}{\tau}\sum_q c_{pq} h_{rs}^{(q)} + \frac{1}{\tau} g_{rs}^{(p)}, \quad r, s = x, y, z. \tag{1.23}$$

In the presence of a constant magnetic field, taking again $b_n = \delta_{nz}$, th equations decouple into three sets of equations. The z–z component is independent of all the others and is unaffected by the magnetic field: it obeys the same equation as (1.23). The x–z and y–z equations are mutually coupled:

$$\partial_t h_{xz}^{(p)} - \Omega h_{yz}^{(p)} = -\frac{1}{\tau}\sum_q c_{pq} h_{xz}^{(q)} + \frac{1}{\tau} g_{xz}^{(p)},$$

$$\partial_t h_{yz}^{(p)} + \Omega h_{xz}^{(p)} = -\frac{1}{\tau}\sum_q c_{pq} h_{yz}^{(q)} + \frac{1}{\tau} g_{yz}^{(p)}. \tag{1.24}$$

Finally, the x–x, x–y and y–y components are mutually coupled *:

$$\partial_t h_{xx}^{(p)} - 2\Omega h_{xy}^{(p)} = -\frac{1}{\tau}\sum_q c_{pq} h_{xx}^{(q)} + \frac{1}{\tau} g_{xx}^{(p)},$$

$$\partial_t h_{yy}^{(p)} - 2\Omega h_{xy}^{(p)} = -\frac{1}{\tau}\sum_q c_{pq} h_{yy}^{(q)} + \frac{1}{\tau} g_{yy}^{(p)},$$

$$\partial_t h_{xy}^{(p)} - \Omega\left(h_{yy}^{(p)} - h_{xx}^{(p)}\right) = -\frac{1}{\tau}\sum_q c_{pq} h_{xy}^{(q)} + \frac{1}{\tau} g_{xy}^{(p)}. \tag{1.25}$$

The determination of the 39 unknown moments has therefore been reduced to a few separate small sets of equations which can be solved, even in presence of a magnetic field, with only a moderate amount of labour.

5.2. *Solution of the linearized moment equations. Asymptotics and Markovianization. Moment description and thermodynamics*

We first consider in this section a particular, simple but non-trivial problem, which we solve explicitly and in great detail. The discussion in this section is, in our view, of utmost importance. It will show *how the hydrodynamic and thermodynamic picture emerges from the kinetic equation, i.e. from its moment representation*. Although the mathematics involved here is utterly simple, the physical interpretation of the results is of major significance.

* We have written six equations for each tensor $h_{rs}^{(p)}$: there are, however, only five independent components, because the tensor is traceless. It will be found that the solution of these equations satisfies the constraint $h_{rr}^{(p)} = 0$ automatically, because the sources are traceless: $g_{rr}^{(p)} = 0$.

We consider the determination of the *parallel components of the electron vector moments in the 13 M approximation*. Setting, as usual, the constant magnetic field in the z-direction, we obtain from (1.7) (with $h_r^{e(5)} \equiv 0$) and (1.21),

$$\dot{h}^{(1)} = -\tau^{-1}\left(c_{11}h^{(1)} + c_{13}h^{(3)}\right) + \tau^{-1}g^{(1)},$$

$$\dot{h}^{(3)} = -\tau^{-1}\left(c_{31}h^{(1)} + c_{33}h^{(3)}\right) + \tau^{-1}g^{(3)}. \tag{2.1}$$

We have abbreviated here, $\tau_e \equiv \tau$, $h_z^{(p)} \equiv h^{(p)}$, $c_{pq}^e \equiv c_{pq}$, and similarly for the source terms g. The latter are taken from (1.12) and (1.13), whereas the collision matrix elements are given in table 4.6.3A. We must keep in mind that the matrix elements c_{11}, c_{33} are positive, and that $c_{13} = c_{31}$.

We have to solve a quite elementary set of ordinary differential equations with constant coefficients. Among the several standard methods, we use here the eigenfunction method [see any textbook on differential equations, e.g. Pontriaguine (1969)]. It is known that the solution of the differential system (2.1) is closely dependent on the algebraic properties of the matrix c_{pq}. The eigenvalues of the latter, λ_i, are the roots of the characteristic equation

$$\left\| -\tau^{-1}c_{pq} - \lambda\delta_{pq} \right\| = 0.$$

It is convenient, as will be seen, to denote the eigenvalues by the notation

$$\lambda_i \equiv -\frac{r_i}{\tau}.$$

The numbers r_i are therefore the roots of

$$r^2 - (c_{11} + c_{33})r + \left(c_{11}c_{33} - c_{13}^2\right) = 0. \tag{2.2}$$

Although this equation is trivially solved, we shall not need the explicit expression of its roots; it is sufficient to know that they satisfy (2.2), which also implies the relations

$$r_1 + r_2 = c_{11} + c_{33} > 0,$$

$$r_1 r_2 = c_{11}c_{33} - c_{13}^2 > 0. \tag{2.3}$$

The important property of these roots is their *positive sign*:

$$r_1 > 0, \qquad r_2 > 0.$$

In order to keep in mind an order of magnitude, we give here their numerical values (which depend on the charge number Z):

$$r_1 = 0.390, \qquad r_2 = 2.476, \qquad [Z = 1],$$

$$r_1 = 0.213, \qquad r_2 = 2.144, \qquad [Z = 10]. \tag{2.4}$$

Having found the eigenvalues, the corresponding eigenvectors, satisfying

$$-r_i \chi_i^{(p)} = -\sum_q c_{pq} \chi_i^{(q)},$$

are easily determined:

$$\chi_i^{(p)} = \left(\beta, \frac{-c_{11} + r_i}{c_{13}} \beta\right),$$

where β is an arbitrary constant. It then follows that the general solution of the homogeneous system (2.1) (with $g^{(p)} = 0$) is

$$h_{[0]}^{(1)}(t) = \alpha_1 \, \mathrm{e}^{-(r_1 t/\tau)} + \alpha_2 \, \mathrm{e}^{-(r_2 t/\tau)},$$

$$h_{[0]}^{(3)}(t) = \alpha_1 \frac{-c_{11} + r_1}{c_{13}} \, \mathrm{e}^{-(r_1 t/\tau)} + \alpha_2 \frac{-c_{11} + r_2}{c_{13}} \, \mathrm{e}^{-(r_2 t/\tau)}, \tag{2.5}$$

where α_1, α_2 are arbitrary constants. In order to solve the inhomogeneous system (2.1), we start from (2.5) and use the classical method of variation of the constants. We find

$$h^{(1)}(t) = \alpha_1 \, \mathrm{e}^{-(r_1 t/\tau)} + \alpha_2 \, \mathrm{e}^{-(r_2 t/\tau)} + \frac{1}{(r_1 - r_2)\tau} \int_0^t \mathrm{d}\theta$$

$$\times \left\{ \mathrm{e}^{-(r_1/\tau)(t-\theta)} \left[(c_{11} - r_2) g^{(1)}(\theta) + c_{13} g^{(3)}(\theta) \right] \right.$$

$$\left. + \mathrm{e}^{-(r_2/\tau)(t-\theta)} \left[(-c_{11} + r_1) g^{(1)}(\theta) - c_{13} g^{(3)}(\theta) \right] \right\}$$

and a similar expression for $h^{(3)}(t)$. We now determine the constants α_1, α_2 in terms of the initial condition,

$$h^{(1)}(0) = \alpha_1 + \alpha_2,$$

$$h^{(3)}(0) = \alpha_1 \frac{-c_{11} + r_1}{c_{13}} + \alpha_2 \frac{-c_{11} + r_2}{c_{13}}.$$

A grouping of all these results allows us to write the *exact solution of the initial value problem* of eq. (2.1) in the compact form

$$h^{(p)}(t) = \sum_{q=1,3} G^{(pq)}(t)\, h^{(q)}(0) + \tau^{-1} \sum_{q=1,3} \int_0^t \mathrm{d}\theta\, G^{(pq)}(\theta)\, g^{(q)}(t-\theta),$$

$$p = 1, 3. \tag{2.6}$$

We have thus explicitly found the *propagator* $G^{(pq)}(t)$ which solves the initial value problem. It is a matrix whose elements are

$$G^{(11)}(t) = \frac{1}{r_1 - r_2}\left((c_{11} - r_2)\, \mathrm{e}^{-r_1(t/\tau)} + (-c_{11} + r_1)\, \mathrm{e}^{-r_2(t/\tau)}\right),$$

$$G^{(13)}(t) = G^{(31)}(t) = \frac{c_{13}}{r_1 - r_2}\left(\mathrm{e}^{-r_1(t/\tau)} - \mathrm{e}^{-r_2(t/\tau)}\right),$$

$$G^{(33)}(t) = \frac{1}{r_1 - r_2}\left((-c_{11} + r_1)\, \mathrm{e}^{-r_1(t/\tau)} + (c_{11} - r_2)\, \mathrm{e}^{-r_2(t/\tau)}\right). \tag{2.7}$$

It is easily seen that

$$G^{(pq)}(0) = \delta_{pq} \tag{2.8}$$

and that these functions satisfy the differential equations

$$\dot{G}^{(pq)}(t) = -\tau^{-1} \sum_m c_{pm}\, G^{(mq)}(t). \tag{2.9}$$

Equation (2.6) contains the complete information included in the 13M linearized equations. It also answers the main problem of transport theory, in providing an expression of the electric current $h_r^{(1)}$ and of the electron heat flux $h_r^{\mathscr{E}(3)}$ in terms of the hydrodynamic and electrodynamic forces contained in the sources $g_r^{(1)}$ and $g_r^{\mathscr{E}(3)}$. This answer is, however, *not* of the "expected" type. There are two typically non-classical features in this solution.

(a) The value of the vector moments $h_r^{(p)}(t)$ at time t is determined by *the whole history of the driving forces* $g_r^{(q)}(t-\theta)$ between time 0 and time t, rather than by the *instantaneous value* $g_r^{(q)}(t)$. When these expressions are substituted into the hydrodynamical equations (**4**.5.3), we obtain an *integro-differential equation for the temperature*. This is a typical *non-Markovian process*: the rate of change of the temperature T_e at time t depends on the whole previous history $T_e(t-\theta)$.

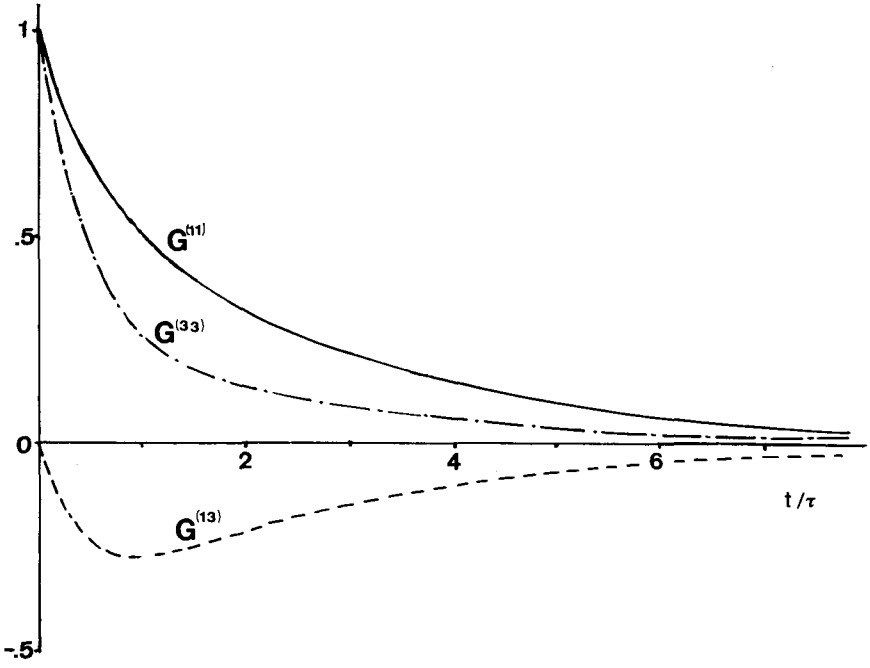


Fig. 2.1. Matrix elements of the propagator $G^{(pq)}(t)$.

(b) The expression of $h^{(p)}(t)$ depends on its initial value $h^{(p)}(0)$. Again, when this term is substituted into the hydrodynamical equations, it implies that the solution of the latter requires the knowledge, not only of the initial data of the hydrodynamical variables, but also of the *initial values of the non-hydrodynamical moments*. These are not specified within the frame-work of hydrodynamics.

We now show that, actually, (2.6) contains information which is irrelevant. Indeed, in deriving the linearized moment equations (2.1), we have made the fundamental assumption that the collisional relaxation time τ_e is much smaller than the hydrodynamical time τ_H, eq. (1.3). We now analyze (2.6) more closely, in order to possibly detect contributions which are small as a result of (1.3).

The starting point of our analysis is the study of the propagator $G^{(pq)}(t)$. It is seen from (2.7) that all its matrix elements are sums of two exponentially decaying terms. The three matrix elements were calculated from the data of table 4.6.3A and eq. (2.7) for $Z = 1$, and plotted in fig. 2.1. One can see from this figure that the initial condition (2.8) is indeed satisfied. But the most interesting feature is the long-time behaviour of $G^{(pq)}(t)$. After a time $t \simeq 7\tau$,

all three matrix elements have decayed to practically zero. But on a macroscopic time scale, such an interval is usually very short; indeed, the macroscopic times are of order τ_H. More precisely, we are interested in hydrodynamics in times t such that

$$\tau \ll t \lesssim \tau_H,$$

or even

$$\tau_H < t. \tag{2.10}$$

Given these facts, it is clear that the first term on the right-hand side of (2.6) goes to zero after a time of order $t \simeq 7\tau$. It represents a rapidly decaying transient. Recalling our qualitative discussion in sections **4**.1 and **4**.2, we can make the arguments quite precise at this point. If we start at time $t = 0$ from an arbitrarily prepared system, the collisions will quickly bring it to a state which depends only on the hydrodynamical quantities (i.e. a "normal" state, close to local equilibrium). In so doing, the collisions wipe out the role of the initial data $h^{(p)}(0)$. In other words, *after a few relaxation times, the system completely forgets the initial values of its non-hydrodynamical moments.*

Let us insist on the following point. The latter sentence *does not* mean that the non-hydrodynamical moments vanish. The system tends towards a state where *the non-hydrodynamical moments become functionals of the hydrodynamical variables.*

Consider now the second term on the right-hand side of (2.6). We see here the propagator $G^{(pq)}(\theta)$ appearing under an integral extended from 0 to t. But actually, this integral only extends effectively from 0 to about 7τ. Over the range $(7\tau, t)$ the integral is zero, because its integrand vanishes. Hence, whenever t is in the range defined by (2.10), the value of the integral is practically independent of its upper limit. We may therefore, without risk of error, replace the upper limit t by ∞: in so doing, we add essentially zero.

As a result of this discussion, (2.6) can be reduced to

$$h^{(p)}(t) = \tau^{-1} \sum_q \int_0^\infty d\theta \, G^{(pq)}(\theta) \, g^{(q)}(t-\theta), \tag{2.11}$$

which is valid under condition (2.10). In other words, (2.11) is an *asymptotic approximation*, valid for times much longer than τ. At this stage, the influence of the initial condition has disappeared. But we may still further simplify this expression.

Relation (2.11) is still non-Markovian, as $h^{(p)}(t)$ depends on the past values of $g^{(q)}(t-\theta)$. However, we now know that the influence of the past

cannot extend indefinitely. Only the values of $g^{(q)}$ comprised between t and $t - 7\tau$ significantly influence $h^{(p)}(t)$: beyond that interval, the factor $G^{(pq)}(\theta)$ vanishes. We now make the following important remark. The propagator is a sharply peaked function (over the long time scale τ_H, it decays from 1 to 0 in the short time 7τ). On the contrary, the hydrodynamical functions $g^{(q)}(t)$ are slowly varying, compared to G. In the time 7τ they change very little from their value at time t. Therefore, over the effective range of variation of G, the sources can be validly represented in terms of a truncated Taylor series,

$$g^{(q)}(t-\theta) \approx g^{(q)}(t) - \theta \dot{g}^{(q)}(t) + \tfrac{1}{2}\theta^2 \ddot{g}^{(q)}(t) + \cdots . \tag{2.12}$$

Substituting this into (2.11) we find

$$h^{(p)}(t) = \sum_q \left[\bar{G}_0^{(pq)} g^{(q)}(t) + \bar{G}_1^{(pq)} \dot{g}^{(q)}(t) + \bar{G}_2^{(pq)} \ddot{g}^{(q)}(t) + \cdots \right], \tag{2.13}$$

where the *constants* $\bar{G}_n^{(pq)}$ are defined as

$$\bar{G}_n^{(pq)} = (-1)^n \frac{1}{n!\tau} \int_0^\infty \mathrm{d}\theta \; \theta^n G^{(pq)}(\theta). \tag{2.14}$$

A simple dimensional argument (or the explicit integration) shows that

$$\bar{G}_n^{(pq)} = \gamma_n^{(pq)} \tau_n, \tag{2.15}$$

where $\gamma_n^{(pq)}$ is a purely numerical constant.

On the other hand, the hydrodynamical source terms are such that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} g^{(q)}(t) \sim \frac{1}{\tau_H^n} g^{(q)}(t).$$

It follows from (2.15) that (2.13) is actually *an expansion of the moment* $h^{(p)}(t)$ *in powers of the small parameter* τ/τ_H. The first term of this expansion is proportional to $g^{(q)} \approx (\tau/\tau_H)$. Thus, in order to be consistent with our linearization, in which we only retained first-order terms, we *must* limit expansion (2.13) to its first term:

$$h^{(p)}(t) = \sum_q \bar{G}^{(pq)} g^{(q)}(t) + \mathrm{O}\left(\tau^2/\tau_H^2\right). \tag{2.16}$$

This expression *defines* the *classical linear transport theory*. Note that we have used the convenient abbreviation

$$\bar{G}^{(pq)} \equiv \bar{G}_0^{(pq)}.$$

The most outstanding feature of this expression is its *Markovian* nature. *The non-hydrodynamical moments* $h^{(p)}(t)$ *appear as simple linear combinations of the sources* $g^{(q)}(t)$ *evaluated at the same time* t. Thus, in going from (2.11) to (2.16), we have performed a *Markovianization* process. The intermediate stage (2.13) represents a so-called *pseudo-Markovian* representation. It expresses $h^{(p)}(t)$ in terms of the sources, as well as of their time derivatives, all evaluated at time t. The presence of the derivative terms is a trace of the fundamentally non-Markovian nature of the problem. Equation (2.13) will be important in a non-linear theory.

The persons who read section **2**.6 and Appendix **2A**.1 will be struck by the analogy between the treatment of the present section and the derivation of the kinetic equation from the **BBGKY** hierarchy. The non-hydrodynamic moments play the role of the correlations, the hydrodynamic variables play the role of the one-particle distribution function. In the kinetic problem we also have two widely separated time scales: the duration of a collision, τ_c, and the relaxation time τ. By exploiting the disparity of these time scales, we went from the exact, non-Markovian master equation (**2**.6.3) (analogous to 2.6) to a Markovian regime where the correlations are functionals of the one-particle distribution function (analogous to 2.16), passing through a pseudo-Markovian stage (**2.A**1.7) (analogous to 2.13). The hydrodynamic problem treated here is the direct continuation of the kinetic theory. In the present problem, the short time scale τ is precisely the *long* time scale of kinetic theory:

$$\tau_c \ll \tau \ll \tau_H. \tag{2.17}$$

Kinetic theory is played in the first range, transport theory is played in the second range.

We now come back to expressions (2.16). Using (2.7), the calculation of the integrals $\bar{G}^{(pq)}$ from (2.14) is trivial, and we find

$$h^{(1)}(t) = \frac{1}{r_1 r_2}\left(c_{33} g^{(1)}(t) - c_{13} g^{(3)}(t)\right),$$

$$h^{(3)}(t) = \frac{1}{r_1 r_2}\left(-c_{13} g^{(1)}(t) + c_{11} g^{(3)}(t)\right). \tag{2.18}$$

Recall that the product $r_1 r_2$ is given by eq. (2.3).

We note that the diagonal elements of this matrix are definite positive numbers (because $c_{11} > 0$, $c_{33} > 0$). This implies the existence of a definite positive electrical conductivity and thermal conductivity, an important property of thermodynamically stable systems. Next, we note the equality of the cross-coefficients (a result of the symmetry $c_{13} = c_{31}$ of the collision matrix). This is a particular expression of the *Onsager symmetry principle* (Onsager 1931, de Groot and Mazur 1984).

Last, but not least, we make a remark of considerable practical importance. We note that the final result (2.18) could have been obtained in a much simpler fashion. If, in the starting equations (2.1), we simply set $\dot{h}^{(p)}(t) = 0$, we find

$$-\tau^{-1}\sum_q c_{pq}h^{(q)} + \tau^{-1}g^{(p)} = 0. \tag{2.19}$$

The solution of this *purely algebraic* set of equations is eq. (2.18). This fact implies an enormous simplification of the procedure.

We now state, without proof, the generalization of our detailed calculation. Consider any subset of moment equations, such as the equations for the parallel components of the vector moments (1.21), or the equations for the transverse components of the vector moments (1.22), etc.

In the former case, the generalization of our results is straightforward: we simply have to consider a larger matrix c_{pq} (e.g. 4×4 in the 29M approximation). The important point is that this matrix still has only real negative eigenvalues.

In the case of the transverse components, the presence of the magnetic terms makes the eigenvalues complex, but these eigenvalues still have a negative real part. As a result, all the elements of the propagator matrices damp out over a time of order τ_α. This property is sufficient for the validity of an asymptotic Markovian solution of the form (2.16) for each subset of moment equations. Moreover, the asymptotic solution is obtained by simply *solving the set of algebraic equations, obtained from the initial set of differential equations by setting all time derivatives equal to zero.*

We now point out an interesting and important point. Consider, for instance, the set of equations for the parallel components of the electron vector moments (the argument being the same for other subsets of moment equations), truncated, say, at the 29M level. The corresponding algebraic equations are (see 2.19)

$$\sum_{q=0}^{3} c_{2p+1,2q+1}h_{\parallel}^{e(2q+1)} = g_{\parallel}^{e(2q+1)}, \quad p = 0, 1, 2, 3. \tag{2.20}$$

The solution of these equations is

$$h_{\|}^{e(2p+1)} = \sum_{q=0}^{3} \bar{G}^{(2p+1,2q+1)} g_{\|}^{e(2q+1)}, \tag{2.21}$$

where $\bar{G}^{(rs)}$ is the inverse of the collision matrix. We thus have a (Markovian) *linear relation between the Hermitian moments* $h_r^{\alpha(n)}$ *and the source terms* $g_r^{\alpha(m)}$. The matrix G is a *symmetric matrix* (being the inverse of the symmetric collision matrix). One would be tempted to identify (2.21) with the *transport equations* or "phenomenological relations" of non-equilibrium thermodynamics (de Groot and Mazur 1984). This interpretation would *not* be correct, as will be presently shown. A precise understanding of this point is illuminating for grasping the nature of the thermodynamic concepts (this problem has, apparently, not been discussed in previous work).

The main point in the argument is the repeatedly stressed fact that the source terms are identically zero for $q \geqslant 2$,

$$g_r^{\alpha(5)} = g_r^{\alpha(7)} = \cdots = 0. \tag{2.22}$$

Therefore, eqs. (2.21), written out explicitly, are

$$h_{\|}^{(1)} = \bar{G}^{(11)} g_{\|}^{(1)} + \bar{G}^{(13)} g_{\|}^{e(3)}, \qquad h_{\|}^{e(3)} = \bar{G}^{(31)} g_{\|}^{(1)} + \bar{G}^{(33)} g_{\|}^{e(3)},$$

$$h_{\|}^{e(5)} = \bar{G}^{(51)} g_{\|}^{(1)} + \bar{G}^{(53)} g_{\|}^{e(3)}, \qquad h_{\|}^{e(7)} = \bar{G}^{(71)} g_{\|}^{(1)} + \bar{G}^{(73)} g_{\|}^{e(3)}. \tag{2.23}$$

The first two equations relate the group of *privileged moments* to the corresponding set of *source terms*. These two equations form, by themselves, a complete set of linear relations, characterized by a 2×2 symmetric matrix ($\bar{G}^{(13)} = \bar{G}^{(31)}$). We have here a one-to-one correspondence with the transport equations of non-equilibrium thermodynamics [de Groot and Mazur 1984, Prigogine 1969). In the latter framework, one defines a set of *fluxes* J_n and a set of *thermodynamic forces* X_n: these are interrelated by a set of *transport equations*

$$J_n = \sum_m L_{nm} X_m. \tag{2.24}$$

The *transport matrix* L_{nm} (whose coefficients are the *transport coefficients*) has the characteristic Onsager symmetry, which reduces here to the simple matrix symmetry $L_{nm} = L_{mn}$. Hence, we can identify the *thermodynamic fluxes* with the *privileged moments* and the *thermodynamic forces* with the *source terms*.

The *non-privileged moments* have an altogether different behaviour. These

moments are exclusively driven by *extraneous forces*: they do not possess any conjugate source terms. The last two equations (2.23) do not have the structure of transport equations (2.24). In particular, the *Onsager symmetry principle does not apply to these moments*. The symmetry is simply irrelevant, because there are no cross-terms associated in pairs. In other words, although the matrix $\bar{G}^{(pq)}$ is a full square matrix, it is applied to a vector $g^{(p)}$ which has only two non-vanishing components.

It should, of course, *not* be concluded that the non-privileged moments are irrelevant in transport theory: they determine the value of the privileged ones indirectly. Indeed, the matrix elements $\bar{G}^{(11)}$, $\bar{G}^{(13)}$, $\bar{G}^{(33)}$ of the *inverse* collision matrix depend, in particular, on matrix elements such as c_{m5}, c_{m7}, ... of the initial matrix. It is for this reason that the accuracy of the thermodynamic transport coefficients is improved when higher (non-privileged) moments are included in the approximation scheme.

As for the higher moments themselves, they are driven by the extraneous forces towards a quasi-steady state defined by (2.23). It turns out that the values of the non-privileged moments in this state are small, compared to those of the privileged ones, and decrease as the order of the moment increases. The relative value of the higher moments is therefore a good measure of the degree of validity of a particular truncation level, as will be seen in section 5.4.

Before concluding this section, we briefly discuss another, even more striking illustration of the difference between privileged and non-privileged moments. It is provided by the higher-order anisotropies, such as the moments $h_{rst}^{\alpha(2p+1)}$ associated with the third-rank irreducible Hermite polynomials $H_{rst}^{(2p+1)}$ (see table **G**1.4.1). *None* of the moments of this type is privileged. Therefore, the asymptotic moment equations are of the form

$$\sum_q c_{2p+1,2q+1} h_{rst}^{\alpha(2q+1)} = 0, \tag{2.25}$$

which have only the trivial solution

$$h_{rst}^{\alpha(2p+1)} = 0. \tag{2.26}$$

It is quite clear that these anisotropies have no thermodynamic interpretation. They exist only as transient quantities, living for a time of order τ_α, after which they decay to zero. Moreover, in a *linear theory*, these higher-order anisotropies (in contrast to the vectorial and second-rank tensorial moments) do not influence at all the thermodynamic transport coefficients. Indeed, their equations of evolution are completely decoupled from the former.

5.3. *The classical transport coefficients*

We now present systematically the results of the classical transport theory. The outcome of this theory is a set of *linear relations connecting the privileged moments* $h_r^{(1)}$, $h_r^{\alpha(3)}$, $h_{rs}^{\alpha(2)}$, *to their conjugate sources*. We shall mainly concentrate on the *21M approximation* (for reasons to be explained), but shall also give the results of the 13M and (in some cases) of the 29M approximation. We assume that there is a *constant, uniform magnetic field* present and that it is directed along the z-axis: $b_n = \delta_{nz}$ *.

A. Electron vector moments

(a) Equations. The basic (algebraic) equations determining the asymptotic values of the Hermitian moments, valid for times much longer than the electron relaxation time τ_e, are obtained from (1.7) by annulling the time derivatives:

$$-\Omega_e \tau_e h_y^{e(p)} + \sum_q c_{pq}^e h_x^{e(q)} = g_x^{e(p)},$$

$$\Omega_e \tau_e h_x^{e(p)} + \sum_q c_{pq}^e h_y^{e(q)} = g_y^{e(p)},$$

$$\sum_q c_{pq}^e h_z^{e(q)} = g_z^{e(p)}, \quad p = 1, 3, 5, (7). \tag{3.1}$$

The electron Larmor frequency Ω_e (a *negative* quantity!) was defined in (1.6), the electron relaxation time τ_e in (4.6.4), and the source terms $g_r^{e(p)}$ in (1.12)–(1.13). We made here the (provisional) convention $h_r^{e(1)} \equiv h_r^{(1)}$, $g_r^{e(1)} \equiv g_r^{(1)}$. Note that all the quantities appearing in these equations are dimensionless. There are two privileged moments in this group: $h_r^{(1)}$, $h_r^{e(3)}$; the higher-order source terms vanish identically: $g_r^{e(5)} = g_r^{e(7)} = 0$.

(b) Solution in dimensionless form. We write the solution of (3.1) for the privileged vector moments in the form

$$h_r^{(1)} = \bar{G}_{rs}^{(11)} g_s^{(1)} + \bar{G}_{rs}^{(13)} g_s^{e(3)},$$

$$h_r^{e(3)} = \bar{G}_{rs}^{(31)} g_s^{(1)} + \bar{G}_{rs}^{(33)} g_s^{e(3)}. \tag{3.2}$$

* The results are approximately valid also if the direction and the intensity of the magnetic field vary very slowly in space. In this case, the reference frame $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ is to be interpreted as a local frame. However, a full discussion of the transport theory in presence of an inhomogeneous and curved magnetic field has to await the treatment of vol. 2.

The coefficients $\overline{G}_{rs}^{(pq)}$ are constants, depending on the magnetic field through the combination $\Omega_e\tau_e$. The following remarkable symmetry relations are found.

(1) For all components $r, s = (x, y, z)$,

$$\overline{G}_{rs}^{(13)} = \overline{G}_{rs}^{(31)}. \tag{3.3}$$

(2) For each couple of moments p, q,

$$\overline{G}_{xx}^{(pq)} = \overline{G}_{yy}^{(pq)}, \qquad \overline{G}_{xy}^{(pq)} = -\overline{G}_{yx}^{(pq)},$$

$$\overline{G}_{xz}^{(pq)} = \overline{G}_{yz}^{(pq)} = \overline{G}_{zx}^{(pq)} = \overline{G}_{zy}^{(pq)} = 0. \tag{3.4}$$

Hence, for each couple (pq), the matrix $\overline{G}_{rs}^{(pq)}$ has exactly *three* independent coefficients. We may therefore write

$$\begin{aligned}
h_x^{(1)} &= \overline{G}_{xx}^{(11)} g_x^{(1)} - \overline{G}_{yx}^{(11)} g_y^{(1)} + \overline{G}_{xx}^{(13)} g_x^{e(3)} - \overline{G}_{yx}^{(13)} g_y^{e(3)}, \\
h_y^{(1)} &= \overline{G}_{yx}^{(11)} g_x^{(1)} + \overline{G}_{xx}^{(11)} g_y^{(1)} + \overline{G}_{yx}^{(13)} g_x^{e(3)} + \overline{G}_{xx}^{(13)} g_y^{e(3)}, \\
h_z^{(1)} &= \overline{G}_{zz}^{(11)} g_z^{(1)} + \overline{G}_{zz}^{(13)} g_z^{e(3)},
\end{aligned} \tag{3.5}$$

and similar relations for $h_r^{e(3)}$. The deeper significance of these symmetry relations will be discussed in detail in section 5.5.

Relations (3.5) can be expressed in a more compact form, independent of the particular reference frame chosen here. In order to do so, we note that, for any vector $\boldsymbol{s} = (s_x, s_y, x_z)$ we have

$$\begin{aligned}
&\boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{s}) = (0, 0, s_z), \\
&\boldsymbol{b}\wedge\boldsymbol{s} = (-s_y, s_x, 0), \\
&\boldsymbol{b}\wedge(\boldsymbol{s}\wedge\boldsymbol{b}) = (s_x, s_y, 0).
\end{aligned} \tag{3.6}$$

We may then rewrite eqs. (3.2), taking account of all the symmetries, in the form

$$\begin{aligned}
\boldsymbol{h}^{(1)} &= \boldsymbol{b}\left[\tilde{\sigma}_{\parallel}(\boldsymbol{g}^{(1)}\cdot\boldsymbol{b}) + \tilde{\alpha}_{\parallel}(\boldsymbol{g}^{e(3)}\cdot\boldsymbol{b})\right] + \boldsymbol{b}\wedge\left(\tilde{\sigma}_{\wedge}\,\boldsymbol{g}^{(1)} + \tilde{\alpha}_{\wedge}\,\boldsymbol{g}^{e(3)}\right) \\
&\quad + \boldsymbol{b}\wedge\left[\tilde{\sigma}_{\perp}(\boldsymbol{g}^{(1)}\wedge\boldsymbol{b}) + \tilde{\alpha}_{\perp}(\boldsymbol{g}^{e(3)}\wedge\boldsymbol{b})\right], \\
\boldsymbol{h}^{e(3)} &= \boldsymbol{b}\left[\tilde{\alpha}_{\parallel}(\boldsymbol{g}^{(1)}\cdot\boldsymbol{b}) + \tilde{\kappa}_{\parallel}^{e}(\boldsymbol{g}^{e(3)}\cdot\boldsymbol{b})\right] + \boldsymbol{b}\wedge\left(\tilde{\alpha}_{\wedge}\,\boldsymbol{g}^{(1)} + \tilde{\kappa}_{\wedge}^{e}\,\boldsymbol{g}^{e(3)}\right) \\
&\quad + \boldsymbol{b}\wedge\left[\tilde{\alpha}_{\perp}(\boldsymbol{g}^{(1)}\wedge\boldsymbol{b}) + \tilde{\kappa}_{\perp}^{e}(\boldsymbol{g}^{e(3)}\wedge\boldsymbol{b})\right].
\end{aligned} \tag{3.7}$$

We have thus introduced *nine* independent dimensionless transport coefficients: three *electrical conductivities* $\tilde{\sigma}_A$, three *electron thermal conductivities* $\tilde{\kappa}^e_A$, and three *thermoelectric coefficients* $\tilde{\alpha}_A$ ($A = \|, \perp, \wedge$). They are easily identified:

$$\tilde{\sigma}_\| = \bar{G}^{(11)}_{zz}, \qquad \tilde{\sigma}_\perp = \bar{G}^{(11)}_{xx}, \qquad \tilde{\sigma}_\wedge = \bar{G}^{(11)}_{yx},$$

and similar relations for $\tilde{\kappa}^e_A$, $\tilde{\alpha}_A$.

(c) Solution in dimensional form. Before giving the explicit expressions of the dimensionless transport coefficients, we transform (3.7) into dimensional relations, using eqs. (4.4.4), (4.3.18), (1.12) and (1.13). We then obtain

$$\boldsymbol{j} = \boldsymbol{b}\left[\sigma_\|(\hat{\boldsymbol{E}}\cdot\boldsymbol{b}) + \alpha_\|(-\nabla T_e)\cdot\boldsymbol{b}\right] + \boldsymbol{b}\wedge\left[\sigma_\wedge\hat{\boldsymbol{E}} + \alpha_\wedge(-\nabla T_e)\right]$$
$$+\boldsymbol{b}\wedge\left[\sigma_\perp(\hat{\boldsymbol{E}}\wedge\boldsymbol{b}) + \alpha_\perp(-\nabla T_e)\wedge\boldsymbol{b}\right], \tag{3.8}$$

$$\boldsymbol{q}^e = \boldsymbol{b}\left[\alpha_\| T_e(\hat{\boldsymbol{E}}\cdot\boldsymbol{b}) + \kappa^e_\|(-\nabla T_e)\cdot\boldsymbol{b}\right] + \boldsymbol{b}\wedge\left[\alpha_\wedge T_e\hat{\boldsymbol{E}} + \kappa^e_\wedge(-\nabla T_e)\right]$$
$$+\boldsymbol{b}\wedge\left[\alpha_\perp T_e(\hat{\boldsymbol{E}}\wedge\boldsymbol{b}) + \kappa^e_\perp(-\nabla T_e)\wedge\boldsymbol{b}\right], \tag{3.9}$$

where the *modified electric field* is defined as

$$\hat{\boldsymbol{E}} = \boldsymbol{E} + \frac{1}{c}(\boldsymbol{u}\wedge\boldsymbol{B}) + \frac{1}{en_e}\nabla(n_eT_e). \tag{3.10}$$

The dimensional transport coefficients are related to the dimensionless ones as

$$\sigma_A = \frac{e^2n_e}{m_e}\tau_e\,\tilde{\sigma}_A, \qquad A = (\|, \wedge, \perp), \tag{3.11}$$

$$\alpha_A = \sqrt{\frac{5}{2}}\,\frac{en_e}{m_e}\tau_e\,\tilde{\alpha}_A, \qquad A = (\|, \wedge, \perp), \tag{3.12}$$

$$\kappa^e_A = \frac{5}{2}\,\frac{n_eT_e}{m_e}\tau_e\,\tilde{\kappa}^e_A, \qquad A = (\|, \wedge, \perp). \tag{3.13}$$

Equations (3.8)–(3.13) represent the final form of the *linear relations between the fluxes* $\boldsymbol{j}$, $\boldsymbol{q}^e$ *and the driving forces* $\hat{\boldsymbol{E}}$, $(-\nabla T_e)$ *in the "classical" transport theory*. The explicit expressions of the dimensionless transport coeffi-

cients are collected in table 3.2, at the end of this section, both in the 13M and in the 21M approximations. They will be discussed in detail in the forthcoming sections.

B. Ion vector moments

(a) Equations. This problem is simpler than the corresponding electron problem. The ion vector moments are not coupled to the electric current $h_r^{(1)}$: this reduces the number of equations to be solved in each approximation. The basic equations are now obtained from (1.9),

$$-\Omega_i \tau_i h_y^{i(p)} + \sum_q c_{pq}^i h_x^{i(q)} = g_x^{i(p)},$$

$$\Omega_i \tau_i h_x^{i(p)} + \sum_q c_{pq}^i h_y^{i(q)} = g_y^{i(p)},$$

$$\sum_q c_{pq}^i h_z^{i(q)} = g_z^{i(p)}, \quad p = 3, 5, (7). \tag{3.14}$$

The ion Larmor frequency Ω_i (a *positive* quantity!) was defined in (1.6), the ion relaxation time τ_i in (4.6.10) and the source terms $g_r^{i(p)}$ in (1.15). There is only one privileged moment in this group: $h_r^{i(3)}$; the higher-order source terms are identically zero: $g_r^{i(5)} = g_r^{i(7)} = 0$.

(b) Solution in dimensionless form. Proceeding as above, we find the simpler relation, replacing (3.7),

$$h_r^{i(3)} = \tilde{\kappa}_\parallel^i \, \boldsymbol{b}(\boldsymbol{b} \cdot \boldsymbol{g}^{i(3)}) + \tilde{\kappa}_\wedge^i (\boldsymbol{b} \wedge \boldsymbol{g}^{i(3)}) + \tilde{\kappa}_\perp^i [\boldsymbol{b} \wedge (\boldsymbol{g}^{i(3)} \wedge \boldsymbol{b})]. \tag{3.15}$$

The *dimensionless ion thermal conductivities* $\tilde{\kappa}_A^i$ are listed in table 3.2 at the end of this section.

(c) Solution in dimensional form. The *ion heat flux* $\boldsymbol{q}^i$ is found to be

$$\boldsymbol{q}^i = -\kappa_\parallel^i \, \boldsymbol{b}(\boldsymbol{b} \cdot \nabla T_i) - \kappa_\wedge^i (\boldsymbol{b} \wedge \nabla T_i) - \kappa_\perp^i [\boldsymbol{b} \wedge ((\nabla T_i) \wedge \boldsymbol{b})]. \tag{3.16}$$

The three independent *ion thermal conductivities* κ_A^i are related to the dimensionless ones by

$$\kappa_A^i = \frac{5}{2} \frac{n_i T_i}{m_i} \tau_i \, \tilde{\kappa}_A^i, \quad A = \parallel, \wedge, \perp. \tag{3.17}$$

C. Electron tensor moments

(a) Equations. As follows from the discussion in section 5.1, eqs. (1.8) (with $\dot{h}^{e(p)}_{rs} = 0$), split into three mutually independent blocks:

$$-2\Omega_e\tau_e h^{e(p)}_{xy} + \sum_q c^e_{pq} h^{e(q)}_{xx} = g^{e(p)}_{xx}, \qquad 2\Omega_e\tau_e h^{e(p)}_{xy} + \sum_q c^e_{pq} h^{e(q)}_{yy} = g^{e(p)}_{yy},$$

$$-\Omega_e\tau_e\left(h^{e(p)}_{yy} - h^{e(p)}_{xx}\right) + \sum_q c^e_{pq} h^{e(q)}_{xy} = g^{e(p)}_{xy},$$

$$-\Omega_e\tau_e h^{e(p)}_{yz} + \sum_q c^e_{pq} h^{e(q)}_{xz} = g^{e(p)}_{xz}, \qquad \Omega_e\tau_e h^{e(p)}_{xz} + \sum_q c^e_{pq} h^{e(q)}_{yz} = g^{e(p)}_{yz},$$

$$\sum_q c^e_{pq} h^{e(q)}_{zz} = g^{e(p)}_{zz}, \quad p = 2, 4, (6). \tag{3.18}$$

There is one privileged moment in this group: $h^{e(2)}_{rs}$; the higher-order source terms vanish identically: $g^{e(4)}_{rs} = g^{e(6)}_{rs} = 0$.

(b) Solution in dimensional form. We directly turn to the dimensional form of the solution. We introduce the usual form of the thermodynamic force (which includes a factor 2) (de Groot and Mazur 1984, Braginskii 1965) by means of the symmetrization operator (**4**.5.28),

$$\nu_{rs} = 2\mathscr{T}_{rs|pq}\ \nabla_p u_q. \tag{3.19}$$

We find, using (**4**.3.17) and (1.14), the following expressions (with $\alpha = e$) for the components of the *dissipative pressure tensor*:

$$\pi^\alpha_{xx} = -\tfrac{1}{2}\left(\eta^\alpha_\parallel + \eta^\alpha_4\right)\nu_{xx} - \tfrac{1}{2}\left(\eta^\alpha_\parallel - \eta^\alpha_4\right)\nu_{yy} + \eta^\alpha_3\,\nu_{xy},$$

$$\pi^\alpha_{yy} = -\tfrac{1}{2}\left(\eta^\alpha_\parallel - \eta^\alpha_4\right)\nu_{xx} - \tfrac{1}{2}\left(\eta^\alpha_\parallel + \eta^\alpha_4\right)\nu_{yy} - \eta^\alpha_3\,\nu_{xy},$$

$$\pi^\alpha_{xy} = -\tfrac{1}{2}\eta^\alpha_3\,\nu_{xx} + \tfrac{1}{2}\eta^\alpha_3\,\nu_{yy} - \eta^\alpha_4\,\nu_{xy}, \qquad \pi^\alpha_{xz} = -\eta^\alpha_2\,\nu_{xz} + \eta^\alpha_1\,\nu_{yz},$$

$$\pi^\alpha_{yz} = -\eta^\alpha_1\,\nu_{xz} - \eta^\alpha_2\,\nu_{yz}, \qquad \pi^\alpha_{zz} = -\eta^\alpha_\parallel\,\nu_{zz}. \tag{3.20}$$

There appear *five* independent *electron viscosity coefficients*, which are of the form

$$\eta^e_B = n_e T_e\,\tau_e\,\tilde{\eta}^e_B, \quad B = \parallel, 1, 2, 3, 4, \tag{3.21}$$

where $\tilde{\eta}^e_B$ are the corresponding dimensionless coefficients, listed in table 3.2 at the end of this section.

Equation (3.20) can also be written in the form *

$$\pi_{rs}^{\alpha} = -\eta_{\parallel}^{\alpha}\, \nu_{rs}^{(\parallel)} - \eta_2^{\alpha}\, \nu_{rs}^{(2)} - \eta_4^{\alpha}\, \nu_{rs}^{(4)} - \eta_1^{\alpha}\, \nu_{rs}^{(1)} - \eta_3^{\alpha}\, \nu_{rs}^{(3)}, \tag{3.22}$$

where we define the tensors

$$\nu^{(\parallel)} = \begin{pmatrix} \tfrac{1}{2}(\nu_{xx}+\nu_{yy}) & 0 & 0 \\ 0 & \tfrac{1}{2}(\nu_{xx}+\nu_{yy}) & 0 \\ 0 & 0 & \nu_{zz} \end{pmatrix}, \qquad \nu^{(2)} = \begin{pmatrix} 0 & 0 & \nu_{xz} \\ 0 & 0 & \nu_{yz} \\ \nu_{xz} & \nu_{yz} & 0 \end{pmatrix},$$

$$\nu^{(4)} = \begin{pmatrix} \tfrac{1}{2}(\nu_{xx}-\nu_{yy}) & \nu_{xy} & 0 \\ \nu_{xy} & -\tfrac{1}{2}(\nu_{xx}-\nu_{yy}) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\nu^{(1)} = \begin{pmatrix} 0 & 0 & -\nu_{yz} \\ 0 & 0 & \nu_{xz} \\ -\nu_{yz} & \nu_{xz} & 0 \end{pmatrix},$$

$$\nu^{(3)} = \begin{pmatrix} -\nu_{xy} & \tfrac{1}{2}(\nu_{xx}-\nu_{yy}) & 0 \\ \tfrac{1}{2}(\nu_{xx}-\nu_{yy}) & \nu_{xy} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

This rather complicated structure of the pressure tensor in a magnetic field will be discussed in section 5.5.

A very remarkable relation between the viscosity coefficients emerges from the calculation (this relation was first discovered by Braginskii 1965):

$$\eta_4^{\alpha}(\Omega_\alpha \tau_\alpha) = \eta_2^{\alpha}(2\Omega_\alpha \tau_\alpha), \qquad \eta_3^{\alpha}(\Omega_\alpha \tau_\alpha) = \eta_1^{\alpha}(2\Omega_\alpha \tau_\alpha). \tag{3.24}$$

This relation holds independently of the level of truncation.

D. Ion tensor moments

The equations for the ion tensor moments are exactly of the same form as (3.18): one merely needs to change all the superscripts e into i: e → i. We then find the same solution as (3.20), with $\alpha = \mathrm{i}$. The relation between the dimen-

* The notation adopted here is different from the one used by Braginskii (1965), in particular in the labelling of the terms. The reason for this relabelling will be justified in section 5.5 (see footnote after eq. (5.49)).

sional *ion viscosity coefficients* and their corresponding dimensionless coefficients, listed in table 3.1, is now

$$\eta_B^{\mathrm{i}} = n_{\mathrm{i}} T_{\mathrm{i}}\, \tau_{\mathrm{i}}\, \tilde{\eta}_B^{\mathrm{i}}, \quad B = \|, 1, 2, 3, 4. \tag{3.25}$$

E. Collected values of the transport coefficients

In order to express the dimensionless transport coefficients in a form as compact as possible, we introduce the following notations:

$$D_{pq}^{\alpha} = c_{pp}^{\alpha} c_{qq}^{\alpha} - c_{pq}^{\alpha 2}, \qquad H_{pqr}^{\alpha} = c_{pr}^{\alpha} c_{qq}^{\alpha} - c_{pq}^{\alpha} c_{qr}^{\alpha},$$

$$S^{\mathrm{e}} = c_{11}^{\mathrm{e}2} + c_{33}^{\mathrm{e}2} + c_{55}^{\mathrm{e}2} + 2c_{13}^{\mathrm{e}2} + 2c_{15}^{\mathrm{e}2} + 2c_{35}^{\mathrm{e}2},$$

$$F_{135}^{\alpha} = c_{55}^{\alpha} D_{13}^{\alpha} - c_{35}^{\alpha} H_{315}^{\alpha} - c_{15}^{\alpha} H_{135}^{\alpha}. \tag{3.26}$$

The magnetic field always enters through the combinations

$$x_{\alpha} = \Omega_{\alpha} \tau_{\alpha} \tag{3.27}$$

[do not forget that $x_{\mathrm{e}} < 0$!]. We introduce the following functions of the magnetic field:

$$G_{13}^{\mathrm{e}}(x_{\mathrm{e}}) = D_{13}^{\mathrm{e}2} + \left(c_{11}^{\mathrm{e}2} + c_{33}^{\mathrm{e}2} + 2c_{13}^{\mathrm{e}2}\right) x_{\mathrm{e}}^2 + x_{\mathrm{e}}^4,$$

$$G_{35}^{\mathrm{i}}(x_{\mathrm{i}}) = D_{35}^{\mathrm{i}2} + \left(c_{33}^{\mathrm{i}2} + c_{55}^{\mathrm{i}2} + 2c_{35}^{\mathrm{i}2}\right) x_{\mathrm{i}}^2 + x_{\mathrm{i}}^4,$$

$$P_{135}^{\mathrm{e}}(x_{\mathrm{e}}) = F_{135}^{\mathrm{e}2} + \left(D_{13}^{\mathrm{e}2} + D_{35}^{\mathrm{e}2} + D_{15}^{\mathrm{e}2} + 2H_{315}^{\mathrm{e}} + 2H_{135}^{\mathrm{e}2} + 2H_{153}^{\mathrm{e}2}\right) x_{\mathrm{e}}^2 + S^{\mathrm{e}} x_{\mathrm{e}}^4 + x_{\mathrm{e}}^6,$$

$$G_{24}^{\alpha}(x_{\alpha}) = D_{24}^{\alpha 2} + \left(c_{22}^{\alpha 2} + c_{44}^{\alpha 2} + 2c_{24}^{\alpha 2}\right) x_{\alpha}^2 + x_{\alpha}^4.$$

For the reader's convenience, we collect in table 3.1 the numerical values of the collision matrix elements (computed from table 4.6.3): these will enable him to quickly calculate the value of any transport coefficient, for *arbitrary values of the charge number Z*. [Note that Braginskii (1965) gives no analytical values of the coefficients, but only numerical values for $Z = 1, 2, 3, 4, \infty$].

The analytic expressions of the dimensionless transport coefficients, evaluated both in the 13M and in the 21M approximations, are collected in

Table 3.1
Numerical values of the collision matrix elements

Electron vector fluxes

$c^e_{11} = 1.0000$ $c^e_{33} = 1.3000 + 0.5657\, Z^{-1}$ $c^e_{55} = 1.5464 + 0.9091\, Z^{-1}$
$c^e_{13} = 0.9487$ $c^e_{35} = -1.3040 - 0.3207\, Z^{-1}$ $c^e_{57} = -1.5703 - 0.8681\, Z^{-1}$
$c^e_{15} = -0.8964$ $c^e_{37} = 1.2730 + 0.1637\, Z^{-1}$ $c^e_{77} = 1.7601 + 3.1659\, Z^{-1}$
$c^e_{17} = 0.8539$

Electron tensor fluxes

$c^e_{22} = 1.2000 + 0.8485\, Z^{-1}$ $c^e_{44} = 1.4571 + 1.0354\, Z^{-1}$
$c^e_{24} = -0.9621 - 0.3402\, Z^{-1}$ $c^e_{46} = -1.3286 - 0.4828\, Z^{-1}$
$c^e_{26} = 0.8018 + 0.1417\, Z^{-1}$ $c^e_{66} = 1.6786 + 1.9177\, Z^{-1}$

Ion vector fluxes

$c^i_{33} = 0.5657$ $c^i_{55} = 0.9091$
$c^i_{35} = -0.3207$ $c^i_{57} = -0.8681$
$c^i_{37} = 0.1637$ $c^i_{77} = 3.1659$

Ion tensor fluxes

$c^i_{22} = 0.8485$ $c^i_{44} = 1.0354$
$c^i_{24} = -0.3402$ $c^i_{46} = -0.4828$
$c^i_{26} = 0.1417$ $c^i_{66} = 1.9177$

table 3.2. These quantities are functions of the dimensionless parameters Z, x_e, x_i. In particular, the parallel transport coefficients only depend on Z.

Finally, we collect in table 3.3 the relations between dimensionless and dimensional transport coefficients.

5.4. *Numerical values of the transport coefficients. Convergence of the approximation scheme*

The results collected in table 3.2, combined with the values of the coefficients given in table 3.1, enables one to calculate immediately the numerical values of all the dimensionless transport coefficients. These depend on two parameters: the charge number Z and the product $x_\alpha = \Omega_\alpha \tau_\alpha$.

In a first group of graphs (figs. 4.1a–d) we plot the *parallel electron transport coefficients* (which are independent of x_e) as functions of Z. (Note that *the dimensionless ion transport coefficients are independent of* Z!). The shape of these curves will be discussed in the next section. Here we want to concentrate on the *convergence of the successive approximations*. The graphs

Table 3.2
Dimensionless transport coefficients.

Coefficient	13M Approximation	21M Approximation
Electrical conductivity		
$\tilde{\sigma}_{\parallel}$	c^{e}_{33}/D^{e}_{13}	D^{e}_{35}/F^{e}_{135}
$\tilde{\sigma}_{\wedge}(x_e)$	$-\frac{x_e}{G^{e}_{13}(x_e)}(c^{e2}_{33}+c^{e2}_{13}+x_e^2)$	$-\frac{x_e}{P^{e}_{135}(x_e)}[(D^{e2}_{35}+H^{e2}_{135}+H^{e2}_{153})+(S^{e}-c^{e2}_{11}-c^{e2}_{13}-c^{e2}_{15})x_e^2+x_e^4]$
$\sigma_{\perp}(x_e)$	$\frac{c^{e}_{33}D^{e}_{13}+c^{e}_{11}x_e^2}{G^{e}_{13}(x_e)}$	$\frac{D^{e}_{35}F^{e}_{135}+(c^{e}_{33}D^{e}_{13}+c^{e}_{55}D^{e}_{15}+2c^{e}_{35}H^{e}_{315})x_e^2+c}{P^{e}_{135}(x_e)}$
Thermoelectric coefficient		
$\tilde{\alpha}_{\parallel}$	$-c^{e}_{13}/D^{e}_{13}$	$-H^{e}_{153}/F^{e}_{135}$
$\tilde{\alpha}_{\wedge}(x_e)$	$\frac{x_e}{G^{e}_{13}(x_e)}c^{e}_{13}(c^{e}_{11}+c^{e}_{33})$	$\frac{x_e}{P^{e}_{135}(x_e)}\{(D^{e}_{15}+D^{e}_{35})H^{e}_{153}+[c^{e}_{13}(c^{e}_{11}+c^{e}_{33})+c^{e}_{15}c^{e}_{35}]x_e^2\}$
$\tilde{\alpha}_{\perp}(x_e)$	$-\frac{c^{e}_{13}(D^{e}_{13}-x_e^2)}{G^{e}_{13}(x_e)}$	$-\{H^{e}_{153}F^{e}_{135}+(c^{e}_{15}H^{e}_{315}+c^{e}_{35}H^{e}_{135}-c^{e}_{55}H^{e}_{153}+c^{e}_{13}D^{e}_{13})x_e^2-x_e^4\}\{P^{e}_{135}(x_e)\}^{-1}$
Electron thermal conductivity		
$\tilde{\kappa}^{e}_{\parallel}$	c^{e}_{11}/D^{e}_{13}	D^{e}_{15}/F^{e}_{135}
$\tilde{\kappa}^{e}_{\wedge}(x_e)$	$-\frac{x_e}{G^{e}_{13}(x_e)}(c^{e2}_{11}+c^{e2}_{13}+x_e^2)$	$-\frac{x_e}{P^{e}_{135}(x_e)}[(D^{e2}_{15}+H^{e2}_{315}+H^{e2}_{153})+(S^{e}-c^{e2}_{13}-c^{e2}_{33}-c^{e2}_{35})x_e^2+x_e^4]$
$\tilde{\kappa}^{e}_{\perp}(x_e)$	$\frac{c^{e}_{11}D^{e}_{13}+c^{e}_{33}x_e^2}{G^{e}_{13}(x_e)}$	$\frac{D^{e}_{15}F^{e}_{135}+(c^{e}_{11}D^{e}_{13}+c^{e}_{55}D^{e}_{35}+2c^{e}_{15}H^{e}_{135})x_e^2+c}{P^{e}_{135}(x_e)}$
Electron viscosity		
$\tilde{\eta}^{e}_{\parallel}$	$1/c^{e}_{22}$	c^{e}_{44}/D^{e}_{24}
$\tilde{\eta}^{e}_{1}(x_e)$	$-\frac{x_e}{c^{e2}_{22}+x_e^2}$	$-\frac{x_e}{G^{e}_{24}(x_e)}(c^{e2}_{44}+c^{e2}_{24}+x_e^2)$
$\tilde{\eta}^{e}_{2}(x_e)$	$\frac{c^{e}_{22}}{c^{e2}_{22}+x_e^2}$	$\frac{c^{e}_{44}D^{e}_{24}+c^{e}_{22}x_e^2}{G^{e}_{24}(x_e)}$
Ion thermal conductivity		
$\tilde{\kappa}^{i}_{\parallel}$	$1/c^{i}_{33}$	c^{i}_{55}/D^{i}_{35}
$\tilde{\kappa}^{i}_{\wedge}(x_i)$	$-\frac{x_i}{c^{i2}_{33}+x_i^2}$	$-\frac{x_i}{G^{i}_{35}(x_i)}(c^{i2}_{55}+c^{i2}_{35}+x_i^2)$
$\tilde{\kappa}^{i}_{\perp}(x_i)$	$\frac{c^{i}_{33}}{c^{i2}_{33}+x_i^2}$	$\frac{c^{i}_{55}D^{i}_{35}+c^{i}_{33}x_i^2}{G^{i}_{35}(x_i)}$

Table 3.2 (continued)

Coefficient	13M Approximation	21M Approximation
Ion viscosity		
$\tilde{\eta}^{\mathrm{i}}_{\parallel}$	$1/c^{\mathrm{i}}_{22}$	$c^{\mathrm{i}}_{44}/D^{\mathrm{i}}_{24}$
$\tilde{\eta}^{\mathrm{i}}_{1}(x_{\mathrm{i}})$	$-\dfrac{x_{\mathrm{i}}}{c^{\mathrm{i}2}_{22}+x_{\mathrm{i}}^2}$	$-\dfrac{x_{\mathrm{i}}}{G^{\mathrm{i}}_{24}(x_{\mathrm{i}})}(c^{\mathrm{i}2}_{44}+c^{\mathrm{i}2}_{24}+x_{\mathrm{i}}^2)$
$\tilde{\eta}^{\mathrm{i}}_{2}(x_{\mathrm{i}})$	$\dfrac{c^{\mathrm{i}}_{22}}{c^{\mathrm{i}2}_{22}+x_{\mathrm{i}}^2}$	$\dfrac{c^{\mathrm{i}}_{44}D^{\mathrm{i}}_{24}+c^{\mathrm{i}}_{22}x_{\mathrm{i}}^2}{G^{\mathrm{i}}_{24}(x_{\mathrm{i}})}$

clearly show that the accuracy of these approximations depends on the particular coefficient considered. Thus, the 13M approximation is quite good for the electrical conductivity and for the thermoelectric coefficient (at small Z). In general, however, *the 13M approximation underestimates most transport coefficients*, especially for high Z. The situation is particularly bad for the electron thermal conductivity and viscosity, where the approximation is off by a factor 2 or 3.

Table 3.3
Relations between dimensional and dimensionless transport coefficients.

Electrical conductivity	$\sigma_A = \dfrac{e^2 n_{\mathrm{e}}}{m_{\mathrm{e}}}\tau_{\mathrm{e}}\,\tilde{\sigma}_A(Z, x_{\mathrm{e}})$
Thermoelectric coefficient	$\alpha_A = \dfrac{e n_{\mathrm{e}}}{m_{\mathrm{e}}}\tau_{\mathrm{e}}\sqrt{\tfrac{5}{2}}\,\tilde{\alpha}_A(Z, x_{\mathrm{e}})$
Electron thermal conductivity	$\kappa^{\mathrm{e}}_A = \dfrac{n_{\mathrm{e}}T_{\mathrm{e}}}{m_{\mathrm{e}}}\tau_{\mathrm{e}}\tfrac{5}{2}\,\tilde{\kappa}^{\mathrm{e}}_A(Z, x_{\mathrm{e}})$
Electron viscosity	$\eta^{\mathrm{e}}_B = n_{\mathrm{e}}T_{\mathrm{e}}\tau_{\mathrm{e}}\,\tilde{\eta}^{\mathrm{e}}_B(Z, x_{\mathrm{e}})$
Ion thermal conductivity	$\kappa^{\mathrm{i}}_A = \dfrac{n_{\mathrm{i}}T_{\mathrm{i}}}{m_{\mathrm{i}}}\tau_{\mathrm{i}}\tfrac{5}{2}\,\tilde{\kappa}^{\mathrm{i}}_A(Z, x_{\mathrm{i}})$
Ion viscosity	$\eta^{\mathrm{i}}_B = n_{\mathrm{i}}T_{\mathrm{i}}\tau_{\mathrm{i}}\,\tilde{\eta}^{\mathrm{i}}_B(Z, x_{\mathrm{i}})$

$$\tau_{\mathrm{e}} = \frac{3}{4\sqrt{2\pi}}\frac{m_{\mathrm{e}}^{1/2}T_{\mathrm{e}}^{3/2}}{Z^2e^4n_{\mathrm{i}}\ln\Lambda}, \qquad \tau_{\mathrm{i}} = \frac{3}{4\sqrt{2\pi}}\frac{m_{\mathrm{i}}^{1/2}T_{\mathrm{i}}^{3/2}}{Z^4e^4n_{\mathrm{i}}\ln\Lambda},$$

$$\ln\Lambda = \begin{cases} 25.3 - 1.15\log n_{\mathrm{i}} + 2.3\log T_{\mathrm{e}}; & T_{\mathrm{e}} \approx T_{\mathrm{i}} > 50\ \mathrm{eV}, \\ \text{eq. (4.6.5)} & \text{in the general case.} \end{cases}$$

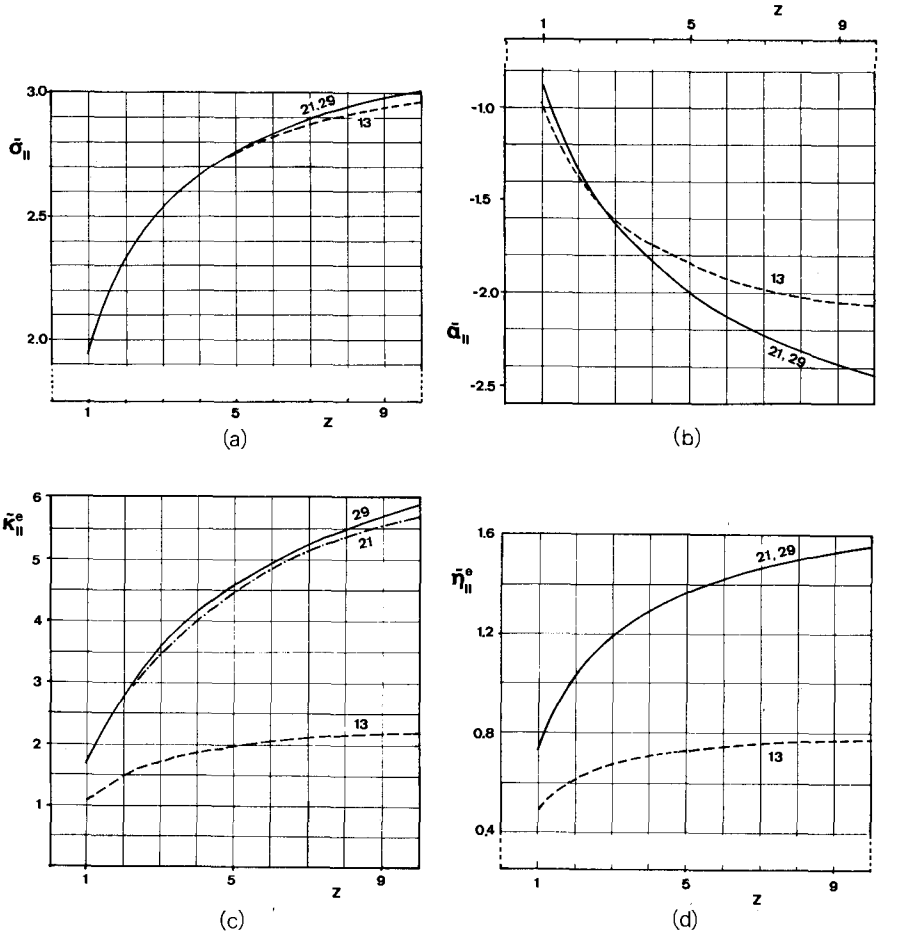


Fig. 4.1. (a) Parallel electrical conductivity; (b) parallel thermoelectric coefficient; (c) parallel electron thermal conductivity; (d) parallel electron viscosity.

A pleasant surprise comes from the fact that the next step, *the 21M approximation provides us already with the practically exact result*. Indeed, we calculated these coefficients in the 29M approximation as well and the results, in most cases, are indistinguishable from the 21M result [see table 4.1]. This point was actually noted already in the old work by Landshoff (1951) and quoted by Spitzer and Härm (1953). The result is confirmed in the similar work by Kaneko (1960), who used 6 Laguerre–Sonine polynomials, i.e. vector Hermite polynomials up to $p = 13$ (for the vector fluxes). Beating his own

Table 4.1
Parallel transport coefficients in three successive approximations ($Z = 1$).

Coefficient	Approximation		
	13M	21M	29M
$\tilde{\sigma}_{\parallel}$	1.932	1.950	1.953
$\tilde{\alpha}_{\parallel}$	−0.982	−0.877	−0.886
$\tilde{\kappa}^{e}_{\parallel}$	1.036	1.659	1.687
$\tilde{\kappa}^{i}_{\parallel}$	1.768	2.210	2.253
$\tilde{\eta}^{e}_{\parallel}$	0.488	0.731	0.731
$\tilde{\eta}^{i}_{\parallel}$	1.178	1.356	1.357

record, Kaneko took up the problem again in two papers (Kaneko and Taguchi 1978, Kaneko and Yamao 1980), where the Chapman–Enskog method was extended to 50 Laguerre–Sonine polynomials! These works confirmed the previous findings and provided transport coefficients accurate to 6 decimal figures, a precision far beyond the needs of experimental plasma physicists.

Figures 4.2a–f show the *transverse components of the transport coefficients*, for $Z = 1$, as a function of x_α. We see again that the 13M approximation is not very accurate for small values of x_α. In general, it does not even reproduce correctly the shape of the curves. However, *for large values of* x_α, the 13M and 21M curves tend to coalesce. This is an indication of an important general result, to be discussed in section 5.6: all the approximate values of the transport coefficients tend, for $x_\alpha > 1$, toward a common value, independent of the approximation.

In order to get an idea about the effect of the charge number Z on the convergence of the approximation for the transverse coefficients, we show in figs. 4.3A, B, two samples of calculations, done in the most extreme case, $Z = \infty$. We see that the perpendicular conductivity $\tilde{\sigma}_{\perp}$ is quite well approximated by its 13M expression over the whole range of x_e. The same statement holds for $\tilde{\sigma}_{\wedge}$ and, to a lesser extent, for the thermoelectric coefficients. On the contrary, the discrepancy between the 13M and the 21M approximations of the thermal conductivity is very badly enhanced as Z increases. This could have been expected from the behaviour of the parallel $\tilde{\kappa}^{e}_{\parallel}$ (fig. 4.1c). The interesting point, however, is that the large discrepancy only occurs for moderate values of x_e. Beyond $x_e = 1$, we see again the convergence of the results toward a common asymptotic value.

An interesting calculation can be done in order to understand the physical origin of the peculiar type of convergence found here. We have discussed till

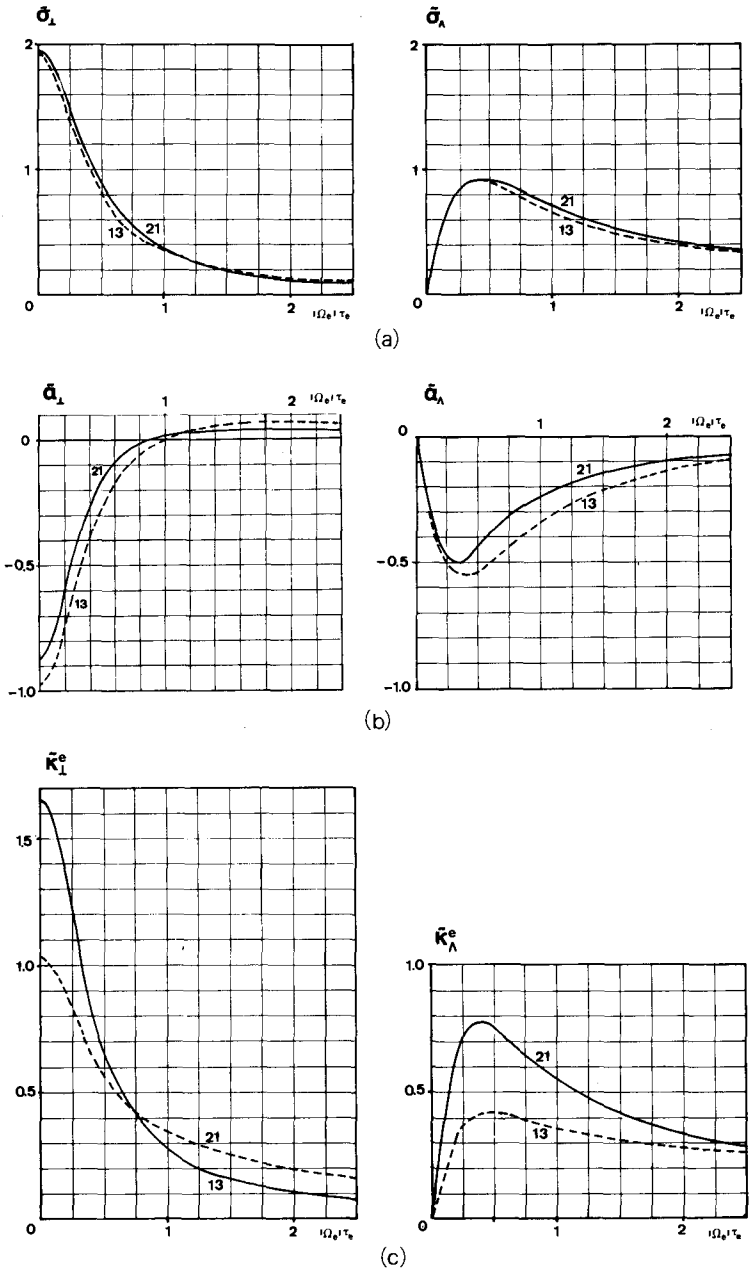
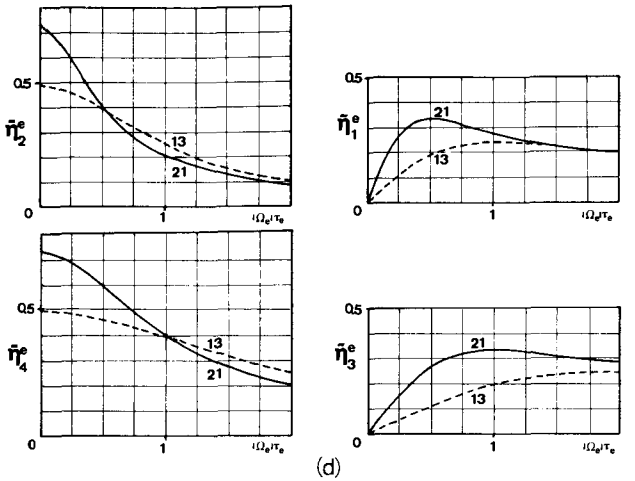
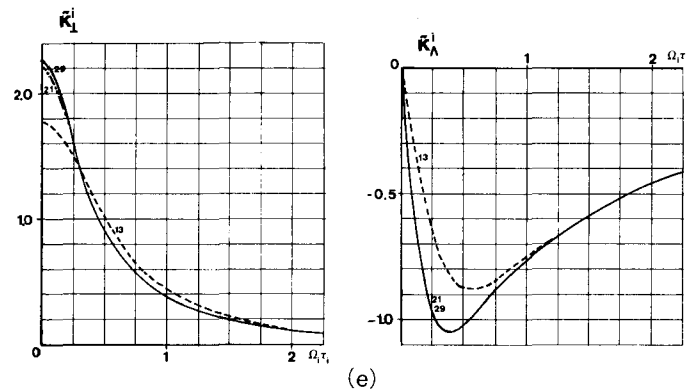


Fig. 4.2. (a) Transverse electrical conductivities ($Z=1$); (b) transverse thermoelectric coefficients ($Z=1$); (c) transverse electron thermal conductivities; (d) transverse electron viscosities ($Z=1$); (e) transverse ion thermal conductivities ($Z=1$); (f) transverse ion viscosities ($Z=1$).



(d)



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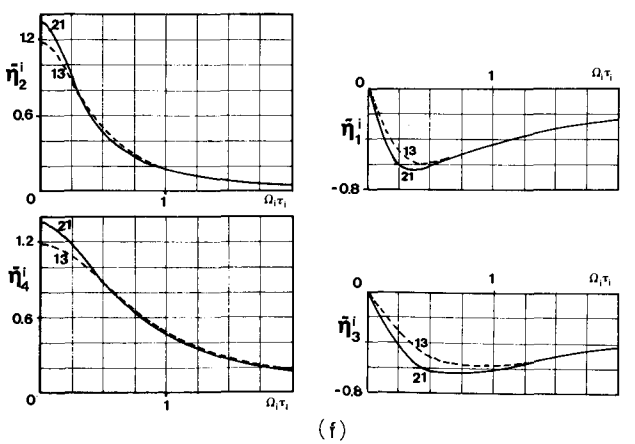

(f)

Fig. 4.2. (continued).

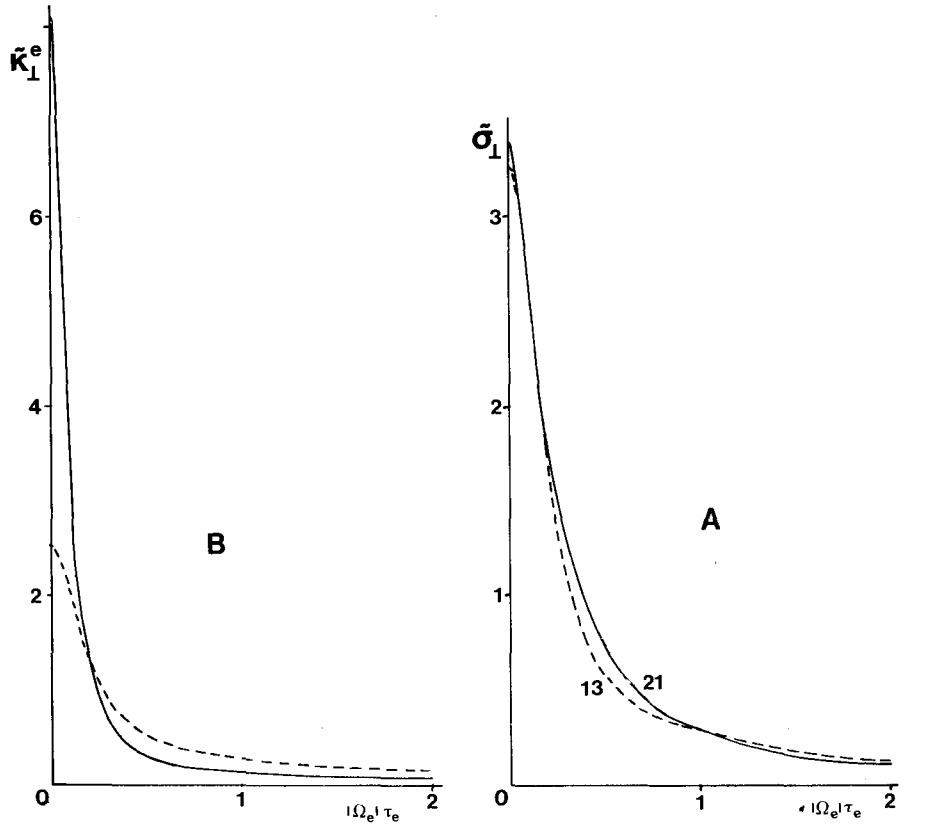

Fig. 4.3. Transverse transport coefficients for $Z = \infty$. A. Perpendicular electrical conductivity. B. Perpendicular electron thermal conductivity.

now only the values of the privileged moments. But we may use our equations for calculating the remaining moments as well. We know from section 5.2 that these are driven by the same forces as the privileged ones. A simple illustration is provided by the tensor moments in absence of a magnetic field. All these moments are of the form

$$h_{rs}^{\alpha(2p)} = \hat{\eta}^{\alpha(2p)} g_{rs}^{\alpha(2p)}, \quad p > 1. \tag{4.1}$$

The coefficients $\hat{\eta}^{\alpha(2p)}$, for $p > 1$, are thus a measure of the size of the moments $h_{rs}^{\alpha(2p)}$ in the quasi-steady state. Table 4.2 gives the values of the coefficients $\hat{\eta}^{\alpha(2p)}$, calculated in the 13M, 21M and 29M approximations. It shows that, for $Z = 1$, the moment $h_{rs}^{\alpha(4)}$ is about 0.5 of $h_{rs}^{\alpha(2)}$, whereas $h_{rs}^{\alpha(6)}$ is

Table 4.2
"Higher-order viscosities" in successive approximations.

Coefficient	Approximation		
	13M	21M	29M
$\hat{\eta}_{\parallel}^{e(2)}$	0.488	0.731	0.731
$\hat{\eta}_{\parallel}^{e(4)}$		0.382	0.383
$\hat{\eta}_{\parallel}^{e(6)}$			0.001
$\hat{\eta}_{\parallel}^{i(2)}$	1.178	1.356	1.357
$\hat{\eta}_{\parallel}^{i(4)}$		0.446	0.452
$\hat{\eta}_{\parallel}^{i(6)}$			0.013

only 1% of the privileged moment (for ions). This clearly explains why the 13M approximation, which neglects $h_{rs}^{\alpha(4)}$, gives such a bad result.

5.5. *Discussion of the transport equations*

The classical theory of transport is characterized by a set of closure relations expressing the fluxes as linear combinations of the driving forces, with constant transport coefficients. Such relations will henceforth be called *transport equations*. This terminology is clearer than the more usual denomination "phenomenological relations". We are here in the domain of *linear non-equilibrium thermodynamics*. All the general principles derived in that field are applicable here (Prigogine 1947, 1969, de Groot and Mazur 1984, Misguich and Balescu 1984).

A. Transport in the absence of a magnetic field

In this case the unperturbed system is *isotropic*. A simple symmetry argument shows that *the transport coefficients are scalars* in this case. The transport equations reduce to

$$\boldsymbol{j} = \sigma_{\parallel}\, \hat{\boldsymbol{E}} + \alpha_{\parallel}\, (-\nabla T_e), \tag{5.1}$$

$$\boldsymbol{q} = \alpha_{\parallel} T_e\, \hat{\boldsymbol{E}} + \kappa_{\parallel}^{e}\, (-\nabla T_e), \tag{5.2}$$

$$\boldsymbol{q}^{i} = \kappa_{\parallel}^{i}\, (-\nabla T_i), \tag{5.3}$$

$$\boldsymbol{\pi}^{\alpha} = \eta_{\parallel}^{\alpha}\, (-\boldsymbol{\nu}). \tag{5.4}$$

Although we discuss the problem of transport without magnetic field, we retain the notations with a "||" subscript. Indeed, we know that all the conclusions reached here remain valid for the *parallel* transport coefficients in the presence of a magnetic field.

It is easily checked from the expressions given in table 3.2 that the following relations hold for $x_\alpha = 0$:

$$\sigma_\perp = \sigma_\parallel, \qquad \sigma_\wedge = 0, \quad [x_e = 0] \tag{5.5}$$

with similar relations for α, κ^α. Moreover,

$$\eta_2^\alpha = \eta_4^\alpha = \eta_\parallel^\alpha, \qquad \eta_1^\alpha = \eta_3^\alpha = 0, \quad [x_\alpha = 0] \tag{5.6}$$

which confirm eqs. (5.1)–(5.4).

Equations (5.1)–(5.4) can be analyzed from the point of view of non-equilibrium thermodynamics. They are a set of linear relations connecting *five dissipative fluxes* ($\boldsymbol{j}$, $\boldsymbol{q}^e$, $\boldsymbol{q}^i$, $\boldsymbol{\pi}^e$, $\boldsymbol{\pi}^i$) to *four thermodynamic forces* ($\hat{\boldsymbol{E}}$, $-\nabla T_e$, $-\nabla T_i$, $-\boldsymbol{\nu}$). Why are there four, rather than five forces? This is because the velocity gradient $\boldsymbol{\nu}$ drives both the electron pressure tensor (or momentum flux) and the ion momentum flux in the same way: only the coefficient of proportionality is different.

In principle, all the fluxes can be coupled to all the forces, thus producing a variety of cross-effects. But actually, there exist "selection rules": certain types of couplings are not allowed. First, a general principle, known as *Curie's principle* (de Groot and Mazur 1984) excludes all linear couplings between fluxes and forces of different tensorial nature. This is confirmed in our kinetic result: the pressure tensors $\boldsymbol{\pi}^\alpha$ are not coupled to the vector forces.

We now discuss the *vector equations* (5.1)–(5.3). The occurrence of two distinct temperatures (hence of two distinct forces ∇T_e, ∇T_i) as well as of two distinct heat fluxes $\boldsymbol{q}^e$, $\boldsymbol{q}^i$ is a peculiar feature of plasma physics, which differs from the usual formulation of the transport equations, in which only one temperature and one global heat flux appear (de Groot and Mazur 1984). The relation between the two descriptions is discussed in detail by Misguich and Balescu (1984). The peculiarity results from the great disparity in the electron and ion masses, $\mu \ll 1$.

This same disparity explains the absence of any coupling between the ion heat flux and the "electronic forces", $\hat{\boldsymbol{E}}$ and ∇T_e. Indeed, the decoupling results from the fact that the ion collision matrix elements c_{pq}^i contain only contributions from the ion–ion collisions, not from the ion–electron collisions, which are smaller by a factor μ (see eq. **4**.6.32). It follows that (5.3) is a purely diagonal transport equation, which is simply the *Fourier law* for the ion heat conduction.

We are now left with eq. (5.1) for the electric current and eq. (5.2) for the electron heat flux. Here appears a real, non-trivial cross-coupling effect: each of the fluxes is driven by *both* forces $\hat{\boldsymbol{E}}$ and $(-\nabla T_e)$ where, in absence of a magnetic field (see 3.10),

$$\hat{\boldsymbol{E}} = \boldsymbol{E} + (en_e)^{-1}\nabla P_e, \quad [B = 0]. \tag{5.7}$$

Simple limiting situations arise in two cases. When the system is *spatially homogeneous*, i.e. when all the gradients vanish, (5.1) reduces to

$$\boldsymbol{j} = \sigma_{\parallel}\, \boldsymbol{E}. \tag{5.8}$$

This is the well-known *Ohm law*. Note, however, that the electric field will also produce a heat flux, because of the non-zero cross-coefficient $\alpha_{\parallel}$,

$$\boldsymbol{q}^e = \alpha_{\parallel} T_e\, \boldsymbol{E}. \tag{5.9}$$

This is related to the *Peltier effect* studied in non-equilibrium thermodynamics of metal thermocouples.

Another simple limiting case arises when there is *no electric field*, $\boldsymbol{E} = 0$, and moreover, the electrons are in *mechanical equilibrium*, i.e. $\nabla P_e = 0$ (or, more generally, when the electric field precisely compensates the electron pressure gradient). In this case, $\hat{\boldsymbol{E}} = 0$ and (5.2) reduces to the ordinary *Fourier law* for the electrons,

$$\boldsymbol{q}^e = -\kappa_{\parallel}^e\, \nabla T_e. \tag{5.10}$$

Note again that in this case, the electron temperature gradient produces an electric current, because of the cross-coefficient $\alpha_{\parallel}$,

$$\boldsymbol{j} = -\alpha_{\parallel}\, \nabla T_e, \tag{5.11}$$

This is the *thermoelectric effect* of non-equilibrium thermodynamics.

In the general case, the two fluxes and the two forces are coupled by a *transport matrix*, that possesses the fundamental Onsager symmetry, which was already previously discussed in section 5.2.

Let us now discuss the values of the transport coefficients from table 3.3. In the absence of a magnetic field, all the dimensionless coefficients $\tilde{L}$ (where L denotes any transport coefficient) are simply numbers, i.e. combinations of the matrix elements c_{pq}. It follows that *all the (dimensional) transport coefficients are proportional to the relaxation time* τ_α. This important feature can be easily understood. It implies that, as the collision frequency τ_α^{-1} increases, the

transport coefficients decrease. In other words, the collisions tend to oppose the transport of matter, momentum and energy: they act as an obstacle to the free flow of these quantities. This intuitive picture would imply that in the limit of no collisions ($\tau_\alpha \to \infty$), the fluxes would become infinite. This conclusion is, however, incorrect, because the assumptions made in this chapter break down when τ_α is too large (see eq. 1.3). The problem of "collisionless transport" must be handled quite differently, as will be seen later.

We now use the results of table 3.3 in order to express the transport coefficients explicitly in terms of the density and the temperatures. If we treat the slowly varying Coulomb logarithm as a constant, we introduce the abbreviation

$$A = \frac{3}{4\sqrt{2\pi}\ln \Lambda},$$

and find

$$\sigma_\parallel = Ae^{-2} m_e^{-1/2} T_e^{3/2} Z^{-1}\, \tilde{\sigma}_\parallel(Z),$$

$$\alpha_\parallel = Ae^{-3} m_e^{-1/2} T_e^{3/2} Z^{-1} \sqrt{\tfrac{5}{2}}\, \tilde{\alpha}_\parallel(Z),$$

$$\kappa_\parallel^\alpha = Ae^{-4} m_\alpha^{-1/2} T_\alpha^{5/2} \left| ZZ_\alpha^3 \right|^{-1} \tfrac{5}{2}\, \tilde{\kappa}_\parallel^\alpha(Z),$$

$$\eta_\parallel^\alpha = Ae^{-4} m_\alpha^{1/2} T_\alpha^{5/2} \left| ZZ_\alpha^3 \right|^{-1} \tilde{\eta}_\parallel^\alpha(Z), \tag{5.12}$$

where we use the obvious notation $Z_e = -1$, $Z_i = Z$.

These formulae show that the *temperature dependence* is particularly important for the thermal conductivities and the viscosities. This implies that the heat equations, obtained by substituting (5.2) into the temperature equations (**4**.5.3), (**4**.5.4) (for $\boldsymbol{u} = \boldsymbol{h}^{\alpha(1)} = 0$) become typical *non-linear equations*:

$$\partial_t T_\alpha = \frac{2}{3n_\alpha} \nabla \cdot \left[\kappa_\parallel^\alpha(T_\alpha)\, \nabla T_\alpha \right], \tag{5.13}$$

with $\kappa_\parallel^\alpha(T_\alpha) \sim T_\alpha^{5/2}$. The type of heat propagation governed by such an equation is quite different from the usual linear heat conduction (see Landau and Lifshitz 1963).

Another remarkable feature is that *the classical transport coefficients do not depend on the density*. This feature is typical of the weak coupling approximation entering the Landau equation. It is actually the only common point between the transport coefficients of a plasma and of a dilute ordinary gas.

An important remark pertains to the relative size of the electron and ion transport coefficients. From (5.12) we obtain

$$\frac{\kappa_{\|}^{\mathrm{i}}}{\kappa_{\|}^{\mathrm{e}}} = \mu^{1/2}\left(\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)^{5/2} Z^{-3}\frac{\tilde{\kappa}_{\|}^{\mathrm{i}}(Z)}{\tilde{\kappa}_{\|}^{\mathrm{e}}(Z)}, \qquad \frac{\eta_{\|}^{\mathrm{i}}}{\eta_{\|}^{\mathrm{e}}} = \mu^{-1/2}\left(\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)^{5/2} Z^{-3}\frac{\tilde{\eta}_{\|}^{\mathrm{i}}(Z)}{\tilde{\eta}_{\|}^{\mathrm{e}}(Z)}. \tag{5.14}$$

The most significant point here is the different appearance of the factor μ. It shows that, most often, the ion thermal conductivity is much smaller than the electron thermal conductivity. On the contrary, the ion viscosity is larger than the electron viscosity. Thus, for $B = 0$, *the energy is mainly transported by the electrons, whereas the momentum is mainly transported by the ions*. The latter part of this statement could, however, become invalid if the electron temperature greatly exceeds the ion temperature, or if the ions are highly charged.

B. Transport in the presence of a constant magnetic field

In this case, the unperturbed plasma is *anisotropic*, hence the transport coefficients are *tensors*, rather than scalars.

Relations (5.1)–(5.4) become

$$j_r = \sigma_{rs}(\boldsymbol{B})\hat{E}_s + \alpha_{rs}(\boldsymbol{B})(-\nabla_s T_{\mathrm{e}}), \tag{5.15}$$

$$q_r^{\mathrm{e}} = \alpha_{rs}(\boldsymbol{B})T_{\mathrm{e}}\,\hat{E}_s + \kappa_{rs}^{\mathrm{e}}(\boldsymbol{B})(-\nabla_s T_{\mathrm{e}}), \tag{5.16}$$

$$q_r^{\mathrm{i}} = \kappa_{rs}^{\mathrm{i}}(\boldsymbol{B})(-\nabla_s T_{\mathrm{i}}), \tag{5.17}$$

$$\pi_{rs}^{\alpha} = \eta_{rsmn}^{\alpha}(\boldsymbol{B})(-\nu_{mn}). \tag{5.18}$$

(a) Vector fluxes. We first discuss eqs. (5.15)–(5.17). The electrical and thermal conductivities and the thermoelectric coefficient are now second-rank tensors. Their structure is, however, not arbitrary: indeed, it is determined by the overall symmetry of the system (de Groot and Mazur 1984).

In absence of the magnetic field, the plasma is isotropic. When the field is switched on, there appears a privileged direction, but the system remains isotropic in the plane perpendicular to $\boldsymbol{B}$. As a result, the transport tensors σ_{rs}, α_{rs}, κ_{rs}^{α} must remain invariant under any rotation around the axis $\boldsymbol{B}$. Let

$\boldsymbol{L}(\boldsymbol{B}) \equiv (L_{rs}(\boldsymbol{B}))$ be any one of these tensors, and let $\boldsymbol{R}$ be a matrix representing an infinitesimal rotation around the z-axis, taken parallel to $\boldsymbol{B}$,

$$\boldsymbol{R} = \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.19}$$

The invariance of the transport tensor is expressed by the conditions

$$L'_{rs}(\boldsymbol{B}) \equiv R_{rm} R_{sn} L_{mn}(\boldsymbol{B}) = L_{rs}(\boldsymbol{B}). \tag{5.20}$$

Performing the transformation explicitly, one easily finds that these conditions imply

$$L_{xx}(\boldsymbol{B}) = L_{yy}(\boldsymbol{B}), \qquad L_{xy}(\boldsymbol{B}) = -L_{yx}(\boldsymbol{B}),$$

$$L_{xz}(\boldsymbol{B}) = L_{yz}(\boldsymbol{B}) = L_{zx}(\boldsymbol{B}) = L_{zy}(\boldsymbol{B}) = 0. \tag{5.21}$$

Thus, the condition of isotropy in a plane perpendicular to $\boldsymbol{B}$ requires that all the transport tensors be of the form

$$\boldsymbol{L} = \begin{pmatrix} L_{\perp}(\boldsymbol{B}) & L_{\wedge}(\boldsymbol{B}) & 0 \\ -L_{\wedge}(\boldsymbol{B}) & L_{\perp}(\boldsymbol{B}) & 0 \\ 0 & 0 & L_{\parallel} \end{pmatrix}. \tag{5.22}$$

Thus, each transport tensor must have exactly three independent elements: $L_{\parallel}$, $L_{\perp}$, $L_{\wedge}$. This requirement agrees perfectly with the results of table 3.2, obtained from kinetic theory.

The non-diagonal transport coefficients give rise to a variety of peculiar phenomena (or "effects"), which have been known for a long time (especially in metals) and are described in books on non-equilibrium thermodynamics. Thus, an electric field in the x-direction can produce a current in the y-direction: this is the *Hall effect*, related to $\sigma_{\wedge}$. A temperature gradient in the x-direction produces a heat flux in the y-direction: this is the *Righi–Leduc effect*, related to $\kappa^{\alpha}_{\wedge}$. Even more interesting are the "doubly-crossed" effects. An electric field along x produces a heat flux along y: the *Nernst effect*; a temperature gradient along x produces an electric current along y: the *Ettingshausen effect*; both are related to $\alpha_{\wedge}$.

Thermodynamics does not require a definite sign for these non-diagonal coefficients (as it does for the diagonal ones, $L_{\parallel}$, $L_{\perp}$): this explains their possible negative sign (see figs. 4.2, b, e). Actually, they enter the transport

matrix (5.22) with both signs. It will be shown in chapter 6 that these phenomena can be called *non-dissipative fluxes* in a quite precise sense.

We now introduce an alternative representation of the transport tensors $\boldsymbol{L}(\boldsymbol{B})$, which has the advantage of being intrinsic (i.e. valid in all reference frames, not only in the particular frame used here). Let us define a right-handed triad of mutually orthogonal unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$, where $\boldsymbol{b}$ is directed along the magnetic field. They obey the usual relation

$$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = \boldsymbol{b}.$$

This triad is of the same kind as the ones introduced in section 1.4. The orientation of the transverse vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ is arbitrary. (This formulation will be easily generalized to the case of an inhomogeneous magnetic field, when the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{b}$ are defined locally.)

We now introduce a *"basis" of nine matrices* which enables us to represent *any* 3×3 matrix as a linear superposition of basis matrices. The idea is very similar to the one used in section 1.6 for the study of 2×2 matrices. We define the following combinations of dyadics:

$$\boldsymbol{S}_1 = \boldsymbol{e}_2\boldsymbol{b} + \boldsymbol{b}\boldsymbol{e}_2, \qquad \boldsymbol{S}_2 = \boldsymbol{b}\boldsymbol{e}_1 + \boldsymbol{e}_1\boldsymbol{b}, \qquad \boldsymbol{S}_3 = \boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1,$$

$$\boldsymbol{S}_4 = \boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2, \qquad \boldsymbol{S}_5 = \boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2, \qquad \boldsymbol{S}_0 = \boldsymbol{b}\boldsymbol{b}, \tag{5.23}$$

and

$$\boldsymbol{A}_1 = \boldsymbol{e}_2\boldsymbol{b} - \boldsymbol{b}\boldsymbol{e}_2, \qquad \boldsymbol{A}_2 = \boldsymbol{b}\boldsymbol{e}_1 - \boldsymbol{e}_1\boldsymbol{b}, \qquad \boldsymbol{A}_3 = \boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_2\boldsymbol{e}_1. \tag{5.24}$$

Clearly, the six matrices $\boldsymbol{S}_n$ are *symmetric*, whereas the three matrices $\boldsymbol{A}_n$ are *antisymmetric*. We quote here a few additional properties, which are easily derived:

$$\boldsymbol{S}_0 + \boldsymbol{S}_4 = \boldsymbol{I}, \tag{5.25}$$

$$\mathrm{Tr}\, \boldsymbol{S}_0 = 1, \qquad \mathrm{Tr}\, \boldsymbol{S}_4 = 2. \tag{5.26}$$

all the other matrices have zero trace. These matrices have the orthogonality property

$$\boldsymbol{S}_m : \boldsymbol{A}_n = 0, \qquad \boldsymbol{S}_m : \boldsymbol{S}_n = (2 - \delta_{n,0})\delta_{mn}. \qquad \boldsymbol{A}_m : \boldsymbol{A}_n = -2\delta_{mn}. \tag{5.27}$$

We now come back to the transport tensor $\boldsymbol{L}$ of eq. (5.22). It is easily seen that it can be represented in an arbitrary frame, as

$$\boldsymbol{L}(\boldsymbol{B}) = L_{\parallel}\, \boldsymbol{bb} + L_{\perp}(B)(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2) + L_{\wedge}(B)(\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_2\boldsymbol{e}_1). \tag{5.28}$$

This is easily translated in terms of the basis matrices,

$$\boldsymbol{L}(\boldsymbol{B}) = L_{\parallel}\, \boldsymbol{S}_0 + L_{\perp}(B)\, \boldsymbol{S}_4 + L_{\wedge}(B)\boldsymbol{A}_3. \tag{5.29}$$

This is *the most general form of a second-rank transport tensor compatible with the requirement of isotropy in the plane perpendicular to* $\boldsymbol{b}$. The interesting and important point is that it only involves *three out of the nine basis matrices*. It manifestly exhibits a symmetric and an antisymmetric part,

$$\boldsymbol{L}(\boldsymbol{B}) = \boldsymbol{L}^{(\mathrm{S})}(\boldsymbol{B}) + \boldsymbol{L}^{(\mathrm{A})}(\boldsymbol{B}), \tag{5.30}$$

with

$$\boldsymbol{L}^{(\mathrm{S})}(\boldsymbol{B}) = L_{\parallel}\, \boldsymbol{S}_0 + L_{\perp}(\boldsymbol{B})\, \boldsymbol{S}_4, \qquad \boldsymbol{L}^{(\mathrm{A})}(\boldsymbol{B}) = L_{\wedge}(\boldsymbol{B})\boldsymbol{A}_3. \tag{5.31}$$

We now consider the *parity of the transport coefficients under an inversion of the magnetic field*: $\boldsymbol{B} \to -\boldsymbol{B}$. This point is a little bit tricky. If we look at the results of table 3.2, we would be tempted to say: all the coefficients depend on $\boldsymbol{B}$ only through the quantity $\Omega_\alpha |\boldsymbol{B}|/m_\alpha c$. Hence, they are all invariant under a change $\boldsymbol{B} \to -\boldsymbol{B}$. But this conclusion is *wrong*. The reason is the peculiar behaviour of the Lorentz force, which is defined through a vector product. The easiest way to obtain the correct transformation rule consists of starting from the initial moment equations (1.7–1.10). They involve the magnetic field *only* through the Lorentz force term,

$$\partial_t h_r^{\alpha(p)} - \Omega_\alpha \varepsilon_{rmn} h_m^{\alpha(p)} b_n = \cdots,$$

where the right-hand side is independent of $\boldsymbol{B}$ (the tensor equations are of a similar form). Changing $\boldsymbol{B}$ to $-\boldsymbol{B}$ is equivalent to the change of the unit vector $\boldsymbol{b} \to -\boldsymbol{b}$,

$$\partial_t h_r^{\alpha(p)} + \Omega_\alpha \varepsilon_{rmn} h_m^{\alpha(p)} b_n = \cdots.$$

This same transformation can be performed by leaving $\boldsymbol{b}$ unchanged, but changing

$$\Omega_\alpha \to -\Omega_\alpha. \tag{5.32}$$

Hence, we get the following practical rule: the magnetic field inversion in the final results, in particular in the transport coefficients, is done by transformation (5.32). We then note the following behaviour, obtained from the results of table 3.2:

$$\begin{cases} L_{\parallel} & \text{is independent of } \Omega_\alpha, \\ L_{\perp}(\Omega_\alpha) & \text{is an even function of } \Omega_\alpha, \\ L_{\wedge}(\Omega_\alpha) & \text{is an odd function of } \Omega_\alpha. \end{cases} \tag{5.33}$$

We may now check the *Onsager symmetry* which, in the presence of a magnetic field, requires for each transport tensor (de Groot and Mazur 1984)

$$L_{rs}(\boldsymbol{B}) = L_{sr}(-\boldsymbol{B}). \tag{5.34}$$

We consider separately the symmetric and the antisymmetric part of each tensor, and perform the field inversion by the rule (5.33),

$$\begin{aligned} L_{rs}^{(\mathrm{S})}(\Omega_\alpha) &= L_{sr}^{(\mathrm{S})}(-\Omega_\alpha) = L_{rs}^{(\mathrm{S})}(-\Omega_\alpha), \\ L_{rs}^{(\mathrm{A})}(\Omega_\alpha) &= L_{sr}^{(\mathrm{A})}(-\Omega_\alpha) = -L_{rs}^{(\mathrm{A})}(-\Omega_\alpha). \end{aligned} \tag{5.35}$$

The first equality is required by the Onsager principle, the second by the parity under matrix transposition. This result is in perfect agreement with the kinetic theory result (5.33).

Of course, the Onsager principle also requires, as before, the identity of the tensor relating $\boldsymbol{j}$ to $(-\nabla T_e)$ and of the tensor relating $\boldsymbol{q}^e$ to $T_e\hat{\boldsymbol{E}}$.

In the limit of a vanishing magnetic field, $\Omega_\alpha \to 0$, the even and odd transport coefficients have a different behaviour:

$$L_{\perp}(B=0) = L_{\parallel}, \qquad L_{\wedge}(B=0) = 0. \tag{5.36}$$

It is clearly seen from (5.22) that the transport tensor then reduces to $L_{\parallel}\boldsymbol{I}$, in agreement with the isotropic transport laws (5.1)–(5.3).

In order to understand the *dependence of the transverse coefficients on the magnetic field intensity*, we must recall the characteristic motion of the charged particles in such a constant field. We know that their motion parallel to the field is unaffected by the latter, whereas their transverse motion is completely impeded. For a very strong magnetic field, the particles would "stick" to the field lines, and there would be no transport in any direction perpendicular to $\boldsymbol{B}$.

This situation is opposed by the collisions: the latter make the particles jump from one field line to another, thus making a transverse transport

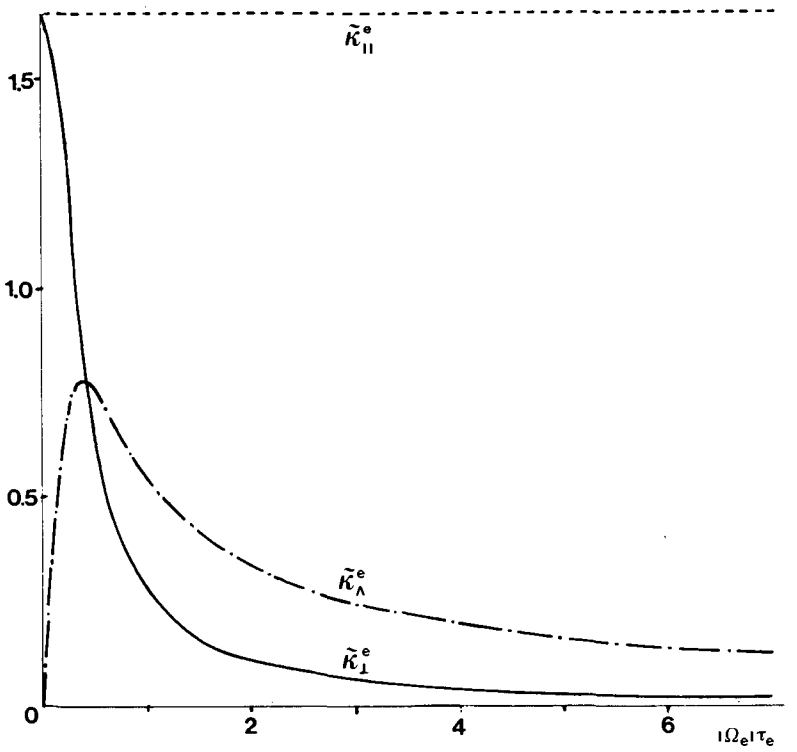

Fig. 5.1. The three electron thermal conductivities (21M; $Z = 1$).

possible. The role of the collisions is therefore completely opposite in the parallel and the transverse directions. In the *parallel* direction, the collisions *impede* the transport, hence $L_{\parallel}$ is an increasing function of τ_α. In the *transverse* direction, the collisions *favour* the transport. Thus, for large values of the parameter $x_\alpha = \Omega_\alpha \tau_\alpha$, i.e. for large magnetic fields and/or small collision frequencies, the coefficients $L_\perp$, $L_\wedge$ are decreasing functions of x_α.

This situation is clearly illustrated in figs. 4.2, as well as in fig. 5.1, which demonstrates the effect by combining the three components of the electron thermal conductivity.

The difference in behaviour between diagonal and non-diagonal coefficients should be stressed. The perpendicular coefficients $L_\perp(x_\alpha)$ are monotonously decreasing functions of $|x_\alpha|$. Starting from the finite value $L_\perp(0) = L_{\parallel}$, they drop rather abruptly to a small value; for $|x_\alpha| > 1$, the variation becomes much slower. The non-diagonal coefficients $L_\wedge$ start from zero and quickly grow to a finite value. In this range, the main effect of the magnetic field is to

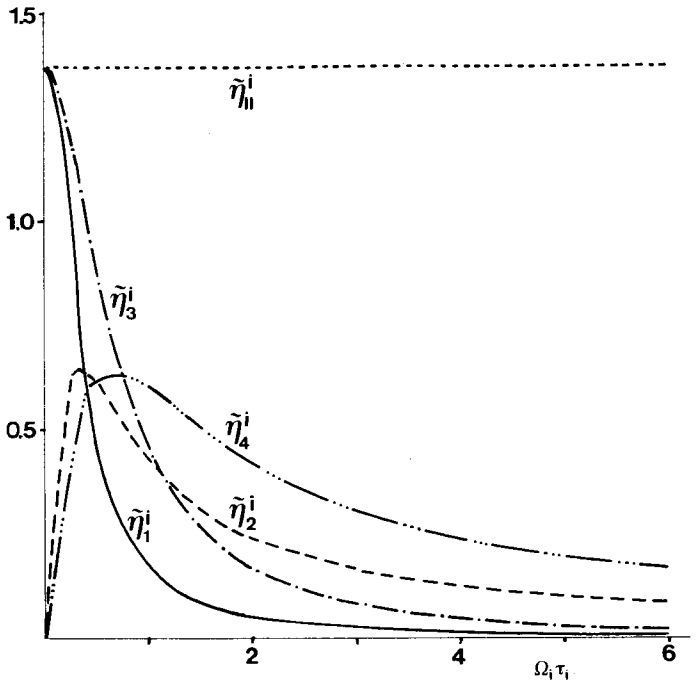

Fig. 5.2. The five ion viscosities. (21M; $Z=1$).

build up the anisotropy, which shows up, in particular, through a Hall effect. For larger values of $|x_\alpha|$, the coefficients $L_\wedge$ start decreasing towards zero, but they do so more slowly than the corresponding $L_\perp$.

It is worth noting that *all* the transport coefficients have the same qualitative behaviour (see figs. 5.1, 5.2). This is an argument showing that the present description is a "fundamental" one, in contrast to other descriptions (such as from Braginskii 1965; see section 5.7) using different sets of fluxes and forces.

Finally, it is interesting to study the effect of the charge number Z on the transverse coefficients, which is illustrated in fig. 5.3. It is seen that the qualitative features described above remain the same, but are considerably enhanced in magnitude as Z increases.

It is rather trivial to note that the ions are much less affected by the magnetic field than the electrons: this is merely a result of their large mass. From eqs. (1.6) and (4.6.11) we find

$$\Omega_i\tau_i = -\mu^{1/2}Z^{-1}(T_i/T_e)^{3/2}\Omega_e\tau_e. \tag{5.37}$$

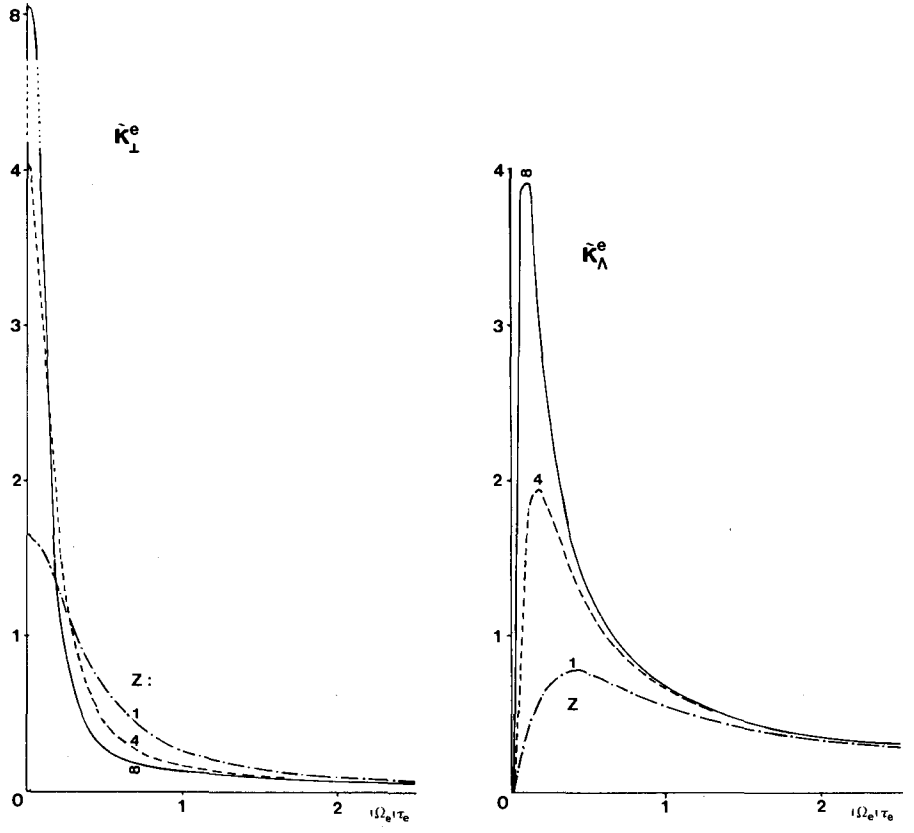

Fig. 5.3. Effect of the charge number Z on the transverse electron thermal conductivities.

As the factors μ, Z^{-1}, T_i/T_e are small (most often $T_i/T_e < 1$), it follows that, for a given magnetic field, $x_i < |x_e|$. As a result, if the graphs for the ion transport coefficients $L^i_\perp$, $L^i_\wedge$ were drawn as functions of x_e, rather than x_i, one would see that the electron transport coefficients die out long before the ion coefficients start decreasing.

We conclude this discussion with the following remark. We have seen that, in the absence of a magnetic field, in a *spatially homogeneous system*, eq. (5.15) reduces to the usual form of the Ohm law (5.8). If we perform the same limit in the presence of a magnetic field, eq. (5.15) reduces to

$$\boldsymbol{j} = \boldsymbol{\sigma} \cdot (\boldsymbol{E} + c^{-1} \boldsymbol{u} \wedge \boldsymbol{B}). \tag{5.38}$$

In other words, the quantity defined by (3.10), which we called the "modified magnetic field", $\hat{\boldsymbol{E}}$, reduces, for a *homogeneous system*, to

$$\hat{\boldsymbol{E}} = \boldsymbol{E} + c^{-1}\boldsymbol{u} \wedge \boldsymbol{B}. \tag{5.39}$$

The driving force for the electric current is this modified field $\hat{\boldsymbol{E}}$, rather than $\boldsymbol{E}$. This is easily understood. A fluid, moving with velocity $\boldsymbol{u}$, does not feel an electric field $\boldsymbol{E}$ as defined in the laboratory reference frame. In order to find the real field acting on the moving plasma element, we must perform Lorentz transformation of the electric field to a frame moving with velocity $\boldsymbol{u}$. The result in the non-relativistic limit is precisely (5.39). This remark is quit important for understanding the equations of magnetohydrodynamics.

(b) Tensor fluxes. We now discuss eqs. (5.18), which have a considerably more complicated structure than the vector equations. This is because the linear transformation from the second-rank tensor ν_{mn} to the second-rank tensor π^{α}_{rs} necessarily involves a fourth-rank tensor η^{α}_{rsmn}. However, the overall symmetry of the problem again considerably reduces the number of coefficients appearing in this fourth-rank tensor. It was shown by Hooyman et al. (1955) (see also de Groot and Mazur, 1984) that the mere requirement of invariance under rotations around an axis parallel to $\boldsymbol{B}$ reduces the number of independent viscosity coefficients to *five*, and requires the form (3.20) (or 3.22–3.23) of the transport equations *. Hence, just like the form (5.22) of the transport matrix in the vector case, the form (3.20) obtained in kinetic theory is required by the general symmetry of the system.

We now show how the fourth-rank transport tensor can be written in an *intrinsic form*. We introduce again the right-handed triad $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$. We then note that when the pressure tensor is written as in (3.22), the component tensors of eq. (3.23) can be identified as

$$\begin{aligned}
\boldsymbol{\nu}^{(\parallel)} &= \tfrac{1}{2}(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2)(\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2) : \boldsymbol{\nu} + \boldsymbol{bb}(\boldsymbol{bb}) : \boldsymbol{\nu}, \\
\boldsymbol{\nu}^{(2)} &= \tfrac{1}{2}(\boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2)(\boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2) : \boldsymbol{\nu} + (\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1)\boldsymbol{e}_1\boldsymbol{e}_2 : \boldsymbol{\nu}, \\
\boldsymbol{\nu}^{(4)} &= (\boldsymbol{e}_1\boldsymbol{b} + \boldsymbol{b}\boldsymbol{e}_1)\boldsymbol{e}_1\boldsymbol{b} : \boldsymbol{\nu} + (\boldsymbol{e}_2\boldsymbol{b} + \boldsymbol{b}\boldsymbol{e}_2)\boldsymbol{e}_2\boldsymbol{b} : \boldsymbol{\nu}, \\
\boldsymbol{\nu}^{(1)} &= (-\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2)\boldsymbol{e}_1\boldsymbol{e}_2 : \boldsymbol{\nu} + (\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1)(\boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2) : \boldsymbol{\nu}, \\
\boldsymbol{\nu}^{(3)} &= -(\boldsymbol{e}_1\boldsymbol{b} + \boldsymbol{b}\boldsymbol{e}_1)\boldsymbol{e}_2\boldsymbol{b} : \boldsymbol{\nu} + (\boldsymbol{e}_2\boldsymbol{b} + \boldsymbol{b}\boldsymbol{e}_2)\boldsymbol{e}_1\boldsymbol{b} : \boldsymbol{\nu}.
\end{aligned} \tag{5.40}$$

We note that all non-diagonal terms, such as $\boldsymbol{e}_1\boldsymbol{e}_2 : \boldsymbol{\nu}$, can be symmetrized as $\frac{1}{2}(\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1) : \boldsymbol{\nu}$ because $\boldsymbol{\nu}$ is a symmetric tensor. We now see that each term

* If there is a bulk viscosity as well, the number becomes seven; but for non-relativistic, weakly coupled plasmas, the bulk viscosity is negligible.

in the expression above simply involves a direct product of two basis matrices $\boldsymbol{S}_n$ or $\boldsymbol{A}_n$ defined in (5.23), (5.24). We may therefore write the transport equation in the form (5.18):

$$\boldsymbol{\pi}^{\alpha} = -\overleftrightarrow{\eta}^{\alpha}(\boldsymbol{B}) : \boldsymbol{\nu}, \tag{5.41}$$

$$\overleftrightarrow{\eta}^{\alpha}(\boldsymbol{B}) = \overleftrightarrow{\eta}^{\alpha(\parallel)} + \overleftrightarrow{\eta}^{\alpha(2)}(\boldsymbol{B}) + \overleftrightarrow{\eta}^{\alpha(4)}(\boldsymbol{B}) + \overleftrightarrow{\eta}^{\alpha(1)}(\boldsymbol{B}) + \overleftrightarrow{\eta}^{\alpha(3)}(\boldsymbol{B}). \tag{5.42}$$

The explicit forms are easily obtained:

$$\overleftrightarrow{\eta}^{\alpha(\parallel)} = \eta_{\parallel}^{\alpha}\left(\boldsymbol{S}_0\boldsymbol{S}_0 + \tfrac{1}{2}\boldsymbol{S}_4\boldsymbol{S}_4\right), \tag{5.43}$$

$$\overleftrightarrow{\eta}^{\alpha(2)}(\boldsymbol{B}) = \tfrac{1}{2}\eta_2^{\alpha}(\boldsymbol{S}_3\boldsymbol{S}_3 + \boldsymbol{S}_5\boldsymbol{S}_5), \tag{5.44}$$

$$\overleftrightarrow{\eta}^{\alpha(4)}(\boldsymbol{B}) = \tfrac{1}{2}\eta_4^{\alpha}(\boldsymbol{S}_1\boldsymbol{S}_1 + \boldsymbol{S}_2\boldsymbol{S}_2), \tag{5.45}$$

$$\overleftrightarrow{\eta}^{\alpha(1)}(\boldsymbol{B}) = \tfrac{1}{2}\eta_1^{\alpha}(\boldsymbol{S}_3\boldsymbol{S}_5 - \boldsymbol{S}_5\boldsymbol{S}_3). \tag{5.46}$$

$$\overleftrightarrow{\eta}^{\alpha(3)}(\boldsymbol{B}) = \tfrac{1}{2}\eta_3^{\alpha}(\boldsymbol{S}_1\boldsymbol{S}_2 - \boldsymbol{S}_2\boldsymbol{S}_1). \tag{5.47}$$

This is the most general form of the viscosity tensor in a uniform magnetic field *.

We note again that not all basis matrices contribute to $\overleftrightarrow{\eta}^{\alpha}$. Indeed, the most general fourth-rank tensor would involve all the $9 \times 9 = 81$ possible pairs of basis matrices; instead, the isotropy requirement reduces this number dramatically to five combinations.

The symmetry properties of the viscosity tensor are not simple. There is, however, one property which is relevant for transport theory; it is the parity under permutation of the two basis matrices in each "super-dyadic". We thus *define* a symmetric and an antisymmetric part of the viscosity tensor as

$$\overleftrightarrow{\eta}^{\alpha}(\boldsymbol{B}) = \overleftrightarrow{\eta}^{\alpha(\mathrm{S})}(\boldsymbol{B}) + \overleftrightarrow{\eta}^{\alpha(\mathrm{A})}(\boldsymbol{B}), \tag{5.48}$$

* Note that Braginskii (1965) also gives an intrinsic representation of the viscosity tensor, which is different from the present one. It is easily checked that his representation is only valid [i.e. reduces to (3.23)] when it applies to a traceless tensor $\boldsymbol{\nu}$. Consider, for instance, his representation of $\boldsymbol{\nu}^{(\parallel)}$ (translated in our notation),

$$\boldsymbol{\nu}^{(\parallel)} = \tfrac{3}{2}\left(\boldsymbol{bb} - \tfrac{1}{3}\boldsymbol{I}\right)\left(\boldsymbol{bb} - \tfrac{1}{3}\boldsymbol{I}\right) : \boldsymbol{\nu}.$$

This is a traceless tensor, whatever the nature of $\boldsymbol{\nu}$. But, for an arbitrary $\boldsymbol{\nu}$, it is not equivalent to the corresponding form (3.23), which is, in general, *not* traceless. If, however, one takes into account the vanishing of Tr $\boldsymbol{\nu}$, the result reduces indeed to (3.23). Similar remarks apply to the other terms of his representation. In conclusion, Braginskii's representation, without being incorrect (because the tensor $\boldsymbol{\nu}$ is indeed traceless), is less general and less transparent than eqs. (5.43)–(5.47).

with

$$\overleftrightarrow{\eta}^{\alpha(\mathrm{S})} = \overleftrightarrow{\eta}^{\alpha(\|)} + \overleftrightarrow{\eta}^{\alpha(2)} + \overleftrightarrow{\eta}^{\alpha(4)},$$

$$\overleftrightarrow{\eta}^{\alpha(\mathrm{A})} = \overleftrightarrow{\eta}^{\alpha(1)} + \overleftrightarrow{\eta}^{\alpha(3)}. \tag{5.49}$$

The corresponding parity is defined as *

$$\eta^{\alpha(\mathrm{S})}_{mnrs} = \eta^{\alpha(\mathrm{S})}_{rsmn}, \qquad \eta^{\alpha(\mathrm{A})}_{mnrs} = -\eta^{\alpha(\mathrm{A})}_{rsmn}. \tag{5.50}$$

Decomposition (5.48) is analogous to decomposition (5.30) of the second-rank transport tensors. It is now easily seen from the results of table 3.2 that

$$\begin{cases} \eta^{\alpha}_{\|} & \text{is independent of} \quad \Omega_\alpha, \\ \eta^{\alpha}_{2p} & \text{are even functions of} \quad \Omega_\alpha, \\ \eta^{\alpha}_{2p+1} & \text{are odd functions of} \quad \Omega_\alpha. \end{cases} \tag{5.51}$$

It is then shown, by a straightforward generalization of the previous discussion, that the *Onsager symmetry principle* takes the form, in the present case,

$$\eta^{\alpha}_{mnrs}(\boldsymbol{B}) = \eta^{\alpha}_{rsmn}(-\boldsymbol{B}), \tag{5.52}$$

which implies

$$\eta^{\alpha(\mathrm{S})}_{mnrs}(\Omega_\alpha) = \eta^{\alpha(\mathrm{S})}_{rsmn}(-\Omega_\alpha) = \eta^{\alpha(\mathrm{S})}_{mnrs}(-\Omega_\alpha),$$

$$\eta^{\alpha(\mathrm{A})}_{mnrs}(\Omega_\alpha) = \eta^{\alpha(\mathrm{A})}_{rsmn}(-\Omega_\alpha) = -\eta^{\alpha(\mathrm{A})}_{mnrs}(-\Omega_\alpha), \tag{5.53}$$

which correspond to relations (5.35).

There is a clear parallelism between the properties of the viscosity tensor and of the second-rank transport tensors. The even coefficients η^{α}_2, η^{α}_4 have properties similar to $L_\perp$, whereas the odd coefficients η^{α}_1, η^{α}_3 behave like $L_\wedge$. In the limit of a vanishing magnetic field we find

$$\eta^{\alpha}_2(0) = \eta^{\alpha}_4(0) = \eta^{\alpha}_{\|}, \qquad \eta^{\alpha}_1(0) = \eta^{\alpha}_3(0) = 0. \tag{5.54}$$

It is then easily seen from (3.23) that the transport equation (5.41) reduces simply to

$$\pi^{\alpha}_{rs} = -\eta^{\alpha}_{\|}\left(\nu^{(\|)}_{rs} + \nu^{(2)}_{rs} + \nu^{(4)}_{s}\right) = -\eta^{\alpha}_{\|}\nu_{rs},$$

which is, indeed, identical to (5.4).

* It is now seen that we have chosen a notation in which the parity of the labels corresponds to the parity of the tensors.

The behaviour for finite fields is understood by the same physical arguments as discussed for the vector fluxes. Figure 5.2 vividly illustrates the analogy between η_2^α, η_4^α and $L_\perp$ on one hand, η_1^α, η_3^α and $L_\wedge$ on the other hand. The latter (odd) contributions to the momentum flux are non-dissipative. These non-dissipative contributions are sometimes called the *gyroviscosity coefficients*. In a strong magnetic field, these coefficients become independent of the relaxation time (see section 5.6). For all these reasons, they are sometimes called the viscosities characterizing a collisionless plasma. This term is, however, misleading, because for finite Ω_α, they do depend on the relaxation time. An interesting discussion of the properties of a "gyroviscous plasma" is given by Newcomb (1966).

5.6. *Limiting values of the transport coefficients in a very strong magnetic field*

It is very interesting to consider separately the values of the transport coefficients in the presence of a very strong magnetic field, i.e. in the limit

$$|x_\alpha| \equiv |\Omega_\alpha|\tau_\alpha \gg 1. \tag{6.1}$$

These asymptotic coefficients describe the "tails" of the curves shown in figs. 4.2, 5.1, 5.2. These simple limiting values have quite interesting intrinsic properties. Moreover, they play a fundamental role in the theory of magnetically confined plasmas.

From the values given in table 3.2, we obtain the following limiting values, by retaining terms up to order $|x_\alpha|^{-2}$:

$$\tilde{\sigma}_\perp^\infty = \frac{c_{11}^e}{(\Omega_e\tau_e)^2}, \qquad \tilde{\sigma}_\wedge^\infty = -\frac{1}{\Omega_e\tau_e},$$

$$\tilde{\alpha}_\perp^\infty = \frac{c_{13}^e}{(\Omega_e\tau_e)^2} \qquad \tilde{\alpha}_\wedge^\infty = 0,$$

$$\tilde{\kappa}_\perp^{\alpha\infty} = \frac{c_{33}^\alpha}{(\Omega_\alpha\tau_\alpha)^2}, \qquad \tilde{\kappa}_\wedge^{\alpha\infty} = -\frac{1}{\Omega_\alpha\tau_\alpha},$$

$$\tilde{\eta}_2^{\alpha\infty} = \frac{c_{22}^\alpha}{(\Omega_\alpha\tau_\alpha)^2}, \qquad \tilde{\eta}_1^{\alpha\infty} = -\frac{1}{\Omega_\alpha\tau_\alpha},$$

$$\tilde{\eta}_4^{\alpha\infty} = \frac{c_{22}^\alpha}{4(\Omega_\alpha\tau_\alpha)^2}, \qquad \tilde{\eta}_3^{\alpha\infty} = -\frac{1}{2\Omega_\alpha\tau_\alpha}. \tag{6.2}$$

A first important fact will be proven now: *the limiting values of the transverse transport coefficients given in eq.* (6.2) *are exact universal values, independent of the approximation scheme* (i.e. of the number of moments retained in their calculation).

This is already suggested by the fact that eqs. (6.2) are obtained either by using the 13M or the 21M expressions of table 3.2. It also appears pictorially in figs. 4.2a–f, where the convergence of the 13M and 21M approximations is evident for large $|x_\alpha|$.

In order to sketch the general proof, we consider the simpler transverse components of the ion vector moments, determined by the first two equations in (3.14). We consider these equations for the moments $h_x^{i(2p+1)}$, $h_y^{i(2p+1)}$ for $p = 1, 2, 3, \ldots, n$. The truncation level, n, is arbitrary. We also remember that $g_r^{i(2p+1)} = 0$ for $p \geqslant 2$.

We are not interested here in the exact solution of (3.14), but only in the leading term, in the limit $|x_i| \gg 1$. We now show that this leading term is easily obtained. (In order to unburden the notation, we use the following symbols in the forthcoming calculation $c_{pq} \equiv c_{pq}^i$, $X \equiv x_i$.)

The solution $h_r^{i(3)}$, for $r = x, y$, is the ratio of two determinants,

$$h_r^{i(3)} = \frac{\Delta_r^{i(3)}}{\Delta}. \tag{6.3}$$

Consider first the denominator; it has the form

$$\Delta = \begin{vmatrix} c_{33} & X & c_{35} & 0 & \cdots & c_{3,2n+1} & 0 \\ -X & c_{33} & 0 & c_{35} & \cdots & 0 & c_{3,2n+1} \\ c_{53} & 0 & c_{55} & X & \cdots & c_{5,2n+1} & 0 \\ 0 & c_{53} & -X & c_{55} & \cdots & 0 & c_{5,2n+1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c_{2n+1,3} & 0 & c_{2n+1,5} & 0 & \cdots & c_{2n+1,2n+1} & X \\ 0 & c_{2n+1,3} & 0 & c_{2n+1,5} & \cdots & -X & c_{2n+1,2n+1} \end{vmatrix}. \tag{6.4}$$

It has a clear structure when decomposed into 2×2 blocks. The blocks along the diagonal are "complete"; they consist of elements c_{pq} along the diagonal, and of elements X off the diagonal. All the other blocks are "incomplete". In order to extract the dominant term in X, one easily convinces himself (e.g. by working out some special cases) that the following practical rule is valid. Consider the columns in succession from the right to the left. In the first column, keep only the dominant element, and annul all the others. In the next one, keep again only the dominant element, provided it is not on the same line as the previous one. Continue in this way up to the leftmost column.

Using this procedure, the dominant contribution to Δ is given by the following determinant, which is easily evaluated,

$$\Delta \simeq \begin{vmatrix} 0 & X & 0 & 0 & \cdots & 0 & 0 \\ -X & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & X & \cdots & 0 & 0 \\ 0 & 0 & -X & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & X \\ 0 & 0 & 0 & 0 & \cdots & -X & 0 \end{vmatrix}.$$

Hence

$$\Delta = X^{2n} + \mathrm{O}(X^{2n-1}). \tag{6.5}$$

The determinant $\Delta_x^{i(3)}$ in (6.3) is obtained from (6.4) by replacing the first column by another column, whose first two elements are $g_x^{i(3)}$, $g_y^{i(3)}$, and all the others are zero. Thus, the expansion of the determinant $\Delta_x^{i(3)}$ leads to

$$\Delta_x^{i(3)} = \delta' g_x^{i(3)} + \delta'' g_y^{i(3)}. \tag{6.6}$$

The determinants δ' and δ'' are evaluated by the same asymptotic analysis as above, with the result

$$\delta' = c_{33} X^{2n-2}, \qquad \delta'' = -X^{2n-1}. \tag{6.7}$$

Hence we find, by combining (6.3)–(6.7) (and reverting to the usual, complete notations),

$$h_x^{i(3)} = \frac{1}{(\Omega_i \tau_i)^2} \left(c_{33}^i g_x^{i(3)} - \Omega_i \tau_i g_y^{i(3)} \right). \tag{6.8}$$

This is exactly the result (6.2).

The fundamental point is that *the level of truncation, n, has dropped out of this expression*. It is seen that the origin of this result is the *vanishing of all source terms for* $p \geqslant 2$.

We have thus obtained expressions (6.2) for $\tilde{\kappa}_\perp^{i\infty}$, $\tilde{\kappa}_\wedge^{i\infty}$ as *general asymptotic values, independent of the level of truncation of the moment equations*. The same analysis can be applied, with the same result (6.2) to all the other transverse limiting transport coefficients.

We now discuss in some detail the *perpendicular (diagonal) transport coefficients*, $\tilde{\sigma}_\perp^\infty$, $\tilde{\alpha}_\perp^\infty$, $\tilde{\kappa}_\perp^{\alpha\infty}$, $\tilde{\eta}_2^{\alpha\infty}$, $\tilde{\eta}_4^{\alpha\infty}$. They all have a very simple dimensionless form

$$\tilde{L}_\perp^{\alpha\infty} = \frac{c_L^\alpha}{(\Omega_\alpha \tau_\alpha)^2}. \tag{6.9}$$

Let us comment on each factor. We have just shown that the values of the coefficients c_L^α are universal; they are simply matrix elements of the collision term c_{pq}^α.

The next important feature is the dependence on $|\Omega_\alpha|^{-2}$: the classical theory predicts an asymptotic decrease of the perpendicular transport coefficients, *proportional to the inverse square of the magnetic field*. Such a strong variation is very favourable for a confinement device, because it keeps the leakage of particles and of energy at a very low level. It was actually one of the main motivations for the start of the big thermonuclear fusion programme around 1950. Unfortunately, these predictions were not borne out by the experiments and the realization of the magnetic fusion turned out to be more difficult than was originally expected. The reason is that many other effects are superposed on the simple, collisional transport mechanism of the classical theory. The resulting effective transport coefficients vary more nearly like B^{-1}, and are therefore less favourable to the confinement. For these reasons, our work does not end at the present chapter: we will have many more things to say about the transport phenomena in plasmas.

The third factor entering (6.9) is the *inverse square of the relaxation time*. This factor determines the dependence on the temperature and the density. In order to validly discuss this point, it is preferable to go over to the dimensional transport coefficients, because the scaling factors between L and $\tilde{L}$ are also functions of temperature and of density.

In the literature on plasma physics, the perpendicular transport coefficients are usually expressed in terms of the *thermal Larmor radius* of the species α, $\rho_{L\alpha}$, defined in (1.5.10), combined with (2.6.13)

$$\rho_{L\alpha}^2 = \frac{V_{T\alpha}^2}{\Omega_\alpha^2} = \left(\frac{m_\alpha c}{e_\alpha B}\right)^2 2\frac{T_\alpha}{m_\alpha}. \tag{6.10}$$

The values of the dimensional perpendicular transport coefficients are collected in table 6.1 and compared to the parallel ones. The comparison immediately reveals a fundamental difference. All the parallel transport coefficients are proportional to the relaxation time, whereas the perpendicular ones are *inversely proportional to the relaxation time* τ_α. Another, very usual way of

Table 6.1
Comparison of the parallel and perpendicular transport coefficients, in the limit of a very strong magnetic field.

Status	Parallel transport	Perpendicular transport
Electrical conductivity	$\sigma_{\parallel} = \dfrac{e^2 n_e}{m_e}\,\tau_e\,\tilde{\sigma}_{\parallel}$	$\sigma_{\perp}^{\infty} = \dfrac{e^2}{2T_e}\,c_{11}^{e}\,\dfrac{n_e\rho_{Le}^2}{\tau_e}$
Thermoelectric coefficient	$\alpha_{\parallel} = \sqrt{\frac{5}{2}}\,\dfrac{en_e}{m_e}\,\tau_e\,\tilde{\alpha}_{\parallel}$	$\alpha_{\perp}^{\infty} = \sqrt{\frac{5}{2}}\,\dfrac{e}{2T_e}\,c_{13}^{e}\,\dfrac{n_e\rho_{Le}^2}{\tau_e}$
Thermal conductivities	$\kappa_{\parallel}^{\alpha} = \dfrac{5}{2}\,\dfrac{n_\alpha T_\alpha}{m_\alpha}\,\tau_\alpha\,\tilde{\kappa}_{\parallel}^{\alpha}$	$\kappa_{\perp}^{\alpha\infty} = \frac{5}{4}\,c_{33}^{\alpha}\,\dfrac{n_\alpha\rho_{L\alpha}^2}{\tau_\alpha}$
Viscosities	$\eta_{\parallel}^{\alpha} = n_\alpha T_\alpha\,\tau_\alpha\,\tilde{\eta}_{\parallel}^{\alpha}$	$\eta_2^{\alpha\infty} = 4\eta_4^{\alpha\infty} = \frac{1}{2}m_\alpha\,c_{22}^{\alpha}\,\dfrac{n_\alpha\rho_{L\alpha}^2}{\tau_\alpha}$

putting this is to use the *collision frequency* $\nu_\alpha = \tau_\alpha^{-1}$. We then see that *the parallel transport coefficients are inversely proportional to the collision frequency*, whereas *the asymptotic perpendicular transport coefficients are proportional to the collision frequency*. This property vividly illustrates our previous discussion: collisions oppose the parallel transport, but favour the perpendicular one. It may be said that in a plasma, in the presence of a constant strong magnetic field, when the collision frequency is increased, the transport of matter, energy and momentum is progressively transferred from the parallel to the perpendicular direction.

We now discuss the dependence of these transport coefficients on temperature and on density. We make the formulae of table 6.1 more explicit by inserting the expressions of the relaxation times (table 5.3). One then finds [with the abbreviation $A \equiv 3/(4\sqrt{2\pi}\,\ln\Lambda)$]

$$\sigma_{\perp}^{\infty} = A^{-1}m_e^{1/2}Z^3e^4c^2B^{-2}n_i^2T_e^{-3/2}\,c_{11}^{e},$$

$$\alpha_{\perp}^{\infty} = A^{-1}m_e^{1/2}Z^3e^3c^2B^{-2}n_i^2T_e^{-3/2}\sqrt{\tfrac{5}{2}}\,c_{13}^{e},$$

$$\kappa_{\perp}^{e\infty} = A^{-1}m_e^{1/2}Z^3e^2c^2B^{-2}n_i^2T_e^{-1/2}\,\tfrac{5}{2}c_{33}^{e},$$

$$\kappa_{\infty\perp}^{i} = A^{-1}m_i^{1/2}Z^2e^2c^2B^{-2}n_i^2T_i^{-1/2}\,\tfrac{5}{2}c_{33}^{i},$$

$$\eta_2^{e\infty} = A^{-1}m_e^{3/2}Z^3e^2c^2B^{-2}n_i^2T_e^{-1/2}\,c_{22}^{e},$$

$$\eta_2^{i\infty} = A^{-1}m_i^{3/2}Z^2e^2c^2B^{-2}n_i^2T_i^{-1/2}\,c_{22}^{i}. \tag{6.11}$$

The coefficients $L_{\perp}^{\infty}$ are thus *proportional to the square of the density*, and are *decreasing functions of temperature*, in striking contrast to $L_{\parallel}$.

Another important difference appears when the perpendicular ion and electron thermal conductivities are compared:

$$\frac{\kappa_{\perp}^{i\infty}}{\kappa_{\perp}^{e\infty}} = \mu^{-1/2} Z^2 \left(\frac{T_i}{T_e}\right)^{-1/2} \frac{c_{33}^{i}}{c_{33}^{e}}. \tag{6.12}$$

Comparing this relation to the corresponding one (5.14) for the parallel thermal conductivities, we see that the factors μ, Z^{-1} (T_i/T_e) appear here with negative powers. Hence, the conclusions of (5.14) are completely reversed. *In the direction perpendicular to the magnetic field, the energy is mainly transported by the ions.* Note again, however, that this conclusion may be reversed if the ion temperature is much higher than the electron temperature.

A similar comparison of the viscosity coefficients yields

$$\frac{\eta_2^{i\infty}}{\eta_2^{e\infty}} = \mu^{-3/2} Z^{-1} \left(\frac{T_i}{T_e}\right)^{-1/2} \frac{c_{22}^{i}}{c_{22}^{e}}. \tag{6.13}$$

Here the conclusion of (5.14) is strongly reinforced by the presence of the very large factor $\mu^{-3/2}$. Thus, *in the perpendicular direction, the momentum is predominantly transported by the ions.*

We now consider the *non-diagonal coefficients* $L_{\wedge}$, which have an altogether different behaviour. We first note that the coefficient $\tilde{\alpha}_{\wedge}^{\infty}$ is much smaller than all the others in a strong magnetic field: it scales like $(\Omega_e \tau_e)^{-3}$. Hence, to the dominant asymptotic order, it can be taken equal to zero. The remaining dimensionless coefficients are all of the form

$$\tilde{L}_{\wedge} = -\frac{1}{\Omega_\alpha \tau_\alpha}. \tag{6.14}$$

This form is again a *universal limiting value*, valid independently of the level of the truncation. A fundamental fact is the *absence of any trace of the collision operator* (i.e. of the matrix elements c_{pq}) in these relations.

Next, we note that these coefficients are larger than the corresponding perpendicular ones by a factor $\Omega_\alpha \tau_\alpha$: they are inversely proportional to B.

The dimensional forms of these coefficients are quite revealing:

$$\sigma_{\wedge}^{\infty} = n_e \frac{c}{eB}, \qquad \alpha_{\wedge}^{\infty} = 0,$$

$$\kappa_{\wedge}^{e\infty} = \tfrac{5}{2} n_e T_e \frac{c}{eB}, \qquad \kappa_{\wedge}^{i\infty} = -\tfrac{5}{2} n_i T_i \frac{c}{ZeB},$$

$$\eta_1^{e\infty} = n_e T_e \frac{m_e c}{eB}, \qquad \eta_1^{i\infty} = -n_i T_i \frac{m_i c}{ZeB}. \tag{6.15}$$

These expressions show that the non-diagonal dimensional transport coefficients are *independent of the collision frequency*. The corresponding fluxes, in the limit of a very strong magnetic field, become *purely non-dissipative*. This point will be further highlighted in the next chapter.

A very shocking feature, at first sight, is the appearance of a *negative sign for the ion thermal conductivity*. It shows again that these fluxes do not conform to the "usual" standards of dissipative transport phenomena. As will be shown, however, in chapter 6, these negative transport coefficients do not imply any contradiction with thermodynamics. It will appear that the non-diagonal components of the fluxes do not produce any entropy. This fact reinforces their interpretation as non-dissipative fluxes.

5.7. *Comparison with other treatments*

The amount of literature published on the classical transport theory is enormous. It would therefore be a very tedious and unnecessary task to perform detailed comparisons with every existing paper. Our purpose in the present section is much more limited: we shall briefly review the various methods used in this field and then focus on a comparison with the works which are commonly used by present-day plasma physicists as "reference works".

Although the problem of classical transport is, in principle, a simple and well-defined subject, many different methods were used in its study, and the results were presented in many different forms. The first aspect is beneficial, as it shows the convergence of the results obtained from different kinetic equations and by different methods of solution of these equations. The second aspect is much less commendable, in as far as each author uses his own definition of the fluxes and of the thermodynamic forces. As a result, it is difficult to compare in detail the various forms of transport equations appearing in the literature. For instance, it is not sufficient to say that the electron thermal conductivity is the proportionality coefficient between the electron heat flux and the electron temperature gradient: its value depends on the definition of all the other fluxes and forces! This point – which will be illustrated below – is well-known in non-equilibrium thermodynamics (de Groot and Mazur 1984). It implies, in practice, that before using a numerical value of a particular transport coefficient, the *complete* context should be checked in order to make sure that it fits the particular situation at hand.

We have collected a certain number of papers on classical transport and give an overview in table 7.1. The rows of this table correspond to the kinetic equation used as a starting point; the columns correspond to the method of solution.

Table 7.1
Selected papers on classical transport theory *.

Kinetic equation	Method of solution	
	Chapman-Enskog	Moments
Boltzmann	Landshoff (1949, 1951) Chapman and Cowling (1952) Kaneko (1960) Ferziger and Kaper (1972) Kaneko and Taguchi (1978) Kaneko and Yamao (1980)	Herdan and Liley (1960)
Landau	Braginskii (1958, 1965) Kaufman (1960, 1966) Robinson and Bernstein (1962) Shkarofsky et al. (1963) Chmielesky and Ferziger (1967) Hinton (1983)	Samokhine (1962, 1963) Kirii and Silin (1969) Silin (1971) Broughan (1982)
BL	Gorbunov and Silin (1964) Rand and Levinsky (1966) Braun (1967)	
Composite	Kihara and Aono (1971) Cohen and Suttorp (1982)	

* The numerical solutions are discussed in the main text.

Four kinetic equations have been used in plasma transport theory. The venerable *Boltzmann equation*, which is the basis of neutral gas kinetic theory, was the first equation used for the calculation of plasma transport coefficients (Chapman and Cowling 1952). Its use with a Coulomb potential leads to well-known divergence difficulties at long distances. On the other hand, the Boltzmann equation is still the basic tool in the theory of weakly ionized gases (which are not discussed in the present book), where the prevalent interactions are non-Coulombic. The *Landau equation* used in this book is the simplest collision model which adequately takes account of the long range of the Coulomb forces (in spite of its divergence difficulties). The *Balescu–Lenard equation* (BL), briefly mentioned in section **2.4**, gives a more correct description of the long-range plasma interactions. Unfortunately, its use in practical calculations is rather difficult. Finally, some *composite kinetic equations* have been proposed independently by Hubbard (1961), Kihara and Aono (1971), Kleinsmith (1976, 1977) and by Mondt (1979). They combine the features of the three previous ones by using some kind of interpolation which ensures the convergence at both short and long distances.

It turns out, fortunately, that *the transport coefficients are rather insensitive to the use of one or another of these kinetic equations*. This fact is particularly well documented in a recent work by Cohen and Suttorp (1982).

The transport coefficients were obtained from an approximate solution of these equations by using (or adapting) either the Chapman–Enskog method or the moment method. To these one could add three works which use a purely numerical solution of the kinetic equations: these are discussed separately below.

The *Chapman–Enskog method* (Chapman and Cowling 1952, Cercignani 1969, Ferziger and Kaper 1972) was historically the first method which proved fully successful for treating the Boltzmann equation (Chapman 1916, 1917, Enskog 1917, 1921). It is a sophisticated asymptotic perturbation method which, unfortunately, leads in practice to a heavy formalism, burdened by quite complicated notations, as can be seen in the references quoted above.

The *moment method* originally put forward by Grad (1949), was only applied till now to very specific problems, or was limited to low approximations. It is one of the purposes of this book to show that the moment method can be developed into a conceptually simple and systematic formalism, allowing one to calculate very precise values of the transport coefficients. A second advantage lies in its unveiling many important structural properties which cannot be easily grasped in the Chapman–Enskog method. Last, but not least, it is very directly applicable to the study of magnetically confined plasmas, as will be seen in the second part of this book: the Chapman–Enskog method cannot be generalized as such for covering these problems.

We now start the more detailed discussion of those papers which are most widely used in the plasma physics literature.

A. Spitzer's theory

This work is contained in two papers: one by Cohen et al. (1950) and one by Spitzer and Härm (1953). It starts from a form of the kinetic equation which is "more or less" equivalent to Landau's viz. The Fokker–Planck equation, derived by Chandrasekhar (1943), linearized around a Maxwellian distribution. The purpose of the first paper was the calculation of the electrical conductivity for a uniform plasma, whereas the second paper combined the electrical conductivity and the thermal conductivity. In both papers, the magnetic field is absent. A stationary solution is sought (in the first case) to the kinetic equation obtained from (3.3.1) with $\Phi^\alpha = 0$, $\partial_t f^\alpha = 0$ and $\mathscr{K}^\alpha$ replaced by its linearized version

$$-\frac{e_\alpha}{m_\alpha} \boldsymbol{E} \cdot \frac{\partial}{\partial \boldsymbol{v}} f_0^\alpha = \mathbb{K}^\alpha (f_0^\alpha \chi_s^\alpha). \tag{7.1}$$

Contrary to most existing works, the kinetic equation was not treated by any expansion method, but was solved directly on a computer. Its solution χ_S^α is still called the *Spitzer function*: it is only known numerically. A similar method was used in the second paper.

This was a very courageous approach in the very early times of computer science. The Spitzer transport coefficients were, for a long time, considered as "exact" reference values. For instance, the earlier versions of the neoclassical theory incorporated the Spitzer function as an essential ingredient, in spite of its being only known numerically (Hinton and Hazeltine 1976).

The absence of an analytical form obscures the structure of the transport theory. We now know, on the other hand, that a very simple analytic approximation, the 21M approximation, yields a remarkable accuracy. As discussed in section 5.4, Kaneko and co-workers (Kaneko 1960, Kaneko and Taguchi 1978, Kaneko and Yamao 1980) verified, by using up to 50 Laguerre–Sonine polynomials, that the increase in accuracy beyond the 21M approximation is minimal, and that the approximate values tend smoothly toward Spitzer's.

This point is also interesting from another point of view. It confirms the insensitiveness of the transport coefficients to the precise kinetic equations used for their calculation. Indeed, Kaneko uses the Boltzmann equation rather than Spitzer's Fokker–Planck equation.

If we now look into the Spitzer–Härm results in more detail, we find a strange feature. The transport equations are presented in the form

$$\boldsymbol{j} = \sigma \boldsymbol{E} + \alpha \nabla T_e, \qquad \boldsymbol{q}^e = -\beta \boldsymbol{E} - K \nabla T_e. \tag{7.2}$$

Comparing these with our equations (5.1), (5.2), we see that the force "conjugate" to $\boldsymbol{j}$ is here the "bare" electric field, instead of the modified field $\hat{\boldsymbol{E}}$, eq. (3.10). This description is thermodynamically incorrect. However, equations of the form (7.2) can be recovered in two *particular* cases.

(1) If it is assumed that the *electron pressure is uniform*, $\hat{\boldsymbol{E}}$ reduces to $\boldsymbol{E}$ (in the absence of a magnetic field) and the coefficients σ, α, β, K of (7.2) reduce to our coefficients $\sigma_\parallel$, $-\alpha_\parallel$, $-\alpha_\parallel T_e$, $\kappa_\parallel^e$.

(2) If it is assumed that the *electron density is uniform*, our equations (5.1), (5.2) can be rewritten as

$$\boldsymbol{j} = \sigma_\parallel \boldsymbol{E} + \left(e^{-1}\sigma_\parallel - \alpha_\parallel\right)\nabla T_e, \qquad \boldsymbol{q}^e = \alpha_\parallel T_e \boldsymbol{E} - \left(\kappa_\parallel^e - e^{-1} T_e \alpha_\parallel\right)\nabla T_e. \tag{7.3}$$

Unfortunately, Spitzer and co-workers are very careless in defining their reference state (they only say that f^0 is a Maxwellian, without giving any formula), so we cannot know a priori which choice is made. We have, however, an indirect piece of evidence. It manifestly appears in (7.3) that, *whenever the*

transport equations are written in terms of fluxes and forces which are not thermodynamically conjugate, the Onsager symmetry does not hold for its coefficient matrix. It is clear, on the other hand, that the Spitzer & Härm coefficients α and β in (7.2) are *unequal*; hence, their results probably correspond to assumption (2). In this case, however, *the coefficient K cannot be identified with the "true" thermal conductivity* $\kappa_{\parallel}^{e}$. This point is a first illustration of the warning given at the beginning of this section. It is indispensable to state very precisely all the physical conditions of the problem in order to determine unambiguously the meaning of each transport coefficient.

B. Braginskii's theory

We now discuss the most widely accepted formulation of the classical transport theory, due to S.I. Braginskii. The original paper was published in 1957 and appeared in an English translation in 1958 (Braginskii 1957). This paper begins with a footnote: "Work performed in 1952". In 1963, Braginskii wrote a detailed review paper in the Russian series *Voprossy Teorii Plazmy, Vol. 1*: this volume was translated into English in 1965 under the title *Reviews of Plasma Physics* (Braginskii 1965). It is the review paper of this volume which became the "bible" of classical plasma transport theory, both in the Soviet Union and in the Western countries.

This work has the following characteristics:

- It treats a two-temperature ion–electron plasma in a constant magnetic field.
- It starts from the Landau equation, simplified by the Lorentz process.
- It uses the Chapman–Enskog (slightly adapted) method with two Laguerre–Sonine polynomials (= 21M) (the parallel coefficients are also given in the 29M approximation).
- It is the first work where *all* the transport coefficients are evaluated and discussed.
- It discusses the connection with non-equilibrium thermodynamics.

We wish to express here our admiration for this careful and valuable work, which deserves its reputation. The methods and results given in the present book incorporate all the ingredients put forward by Braginskii 35 years ago. We shall, however, formulate some remarks about his results.

First, we note that the presentation of his results is very different from ours for the *electron vector fluxes*. Instead of writing transport equations connecting the fluxes ($\boldsymbol{j}, \boldsymbol{q}^{e}$) to the forces ($\hat{\boldsymbol{E}}, \nabla T_e$), he chooses as fluxes ($\boldsymbol{R}^{ei}, \boldsymbol{q}^{e}$) and as forces ($\boldsymbol{j}, \nabla T_e$). This choice seems rather unnatural, because the friction force $\boldsymbol{R}^{ei}$ is not a readily measurable quantity, and $\boldsymbol{j}$ is, intuitively, a flux rather than a force. Our own choice is "natural" in the sense that it comes out directly from the hydrodynamical balance equations (**4**.5.12), (**4**.5.22)–(**4**.5.24) without any additional manipulations. (Hydrodynamics does not provide us

with a balance equation for the friction!) Moreover, as will be seen in Part II (chs. 13, 15–18) in this book, the formulation chosen here is quite well adapted to the neoclassical generalization; this is not the case for Braginskii's formulation.

In order to compare Braginskii's results to ours, we need to perform a linear transformation. In *dimensionless form*, Braginskii's transport equations are written as relations connecting the dimensionless friction $\tau_e \boldsymbol{Q}^{(1)}$ and the heat flux $\boldsymbol{h}^{e(3)}$ to the dimensionless current $\boldsymbol{h}^{(1)}$ and temperature gradient $\boldsymbol{g}^{e(3)}$:

$$\tau_e \boldsymbol{Q}^{(1)} = -\tilde{\rho} \cdot \boldsymbol{h}^{(1)} + \tilde{\boldsymbol{b}} \cdot \boldsymbol{g}^{e(3)}, \qquad \boldsymbol{h}^{e(3)} = \tilde{\boldsymbol{b}}' \cdot \boldsymbol{h}^{(1)} + \tilde{\kappa}'^{e} \cdot \boldsymbol{g}^{e(3)}, \tag{7.4}$$

where ρ, $\boldsymbol{b}$, $\boldsymbol{b}'$, $\kappa'^{(e)}$ are tensors represented by 3×3 matrices. Our equations (5.15)–(5.16) are written as

$$\boldsymbol{h}^{(1)} = \tilde{\sigma} \cdot \boldsymbol{g}^{(1)} + \tilde{\alpha} \cdot \boldsymbol{g}^{e(3)}, \qquad \boldsymbol{h}^{e(3)} = \tilde{\alpha} \cdot \boldsymbol{g}^{(1)} + \tilde{\kappa}^{e} \cdot \boldsymbol{g}^{e(3)}. \tag{7.5}$$

In order to relate these equations to (7.4), we need to adjoin the expression of the friction force, taken from (**4**.6.34),

$$\tau_e \boldsymbol{Q}^{(1)} = -c_{11}^{e} \boldsymbol{h}^{(1)} - c_{13}^{e} \boldsymbol{h}^{e(3)} - c_{15}^{e} \boldsymbol{h}^{e(5)} - \cdots. \tag{7.6}$$

The problem consists of eliminating $\boldsymbol{g}^{(1)}$ from these equations. *This elimination process will be different in the various approximations* (13M, 21M, $\cdots$). In particular, we see that beyond the 13M approximation, we need additional equations, relating the *non-privileged moments* to the sources $\boldsymbol{g}^{(1)}$, $\boldsymbol{g}^{e(3)}$, for instance,

$$\boldsymbol{h}^{e(5)} = \tilde{\beta} \cdot \boldsymbol{g}^{(1)} + \tilde{\gamma} \cdot \boldsymbol{g}^{e(3)}. \tag{7.7}$$

These relations introduce "pseudo-transport coefficients" $\tilde{\beta}$, $\tilde{\gamma}$, which are easily obtained by solving eqs. (3.1) in the relevant approximation.

We then obtain (in the 21M approximation) from the first eq. (7.5)

$$\boldsymbol{g}^{(1)} = \tilde{\sigma}^{-1} \cdot \boldsymbol{h}^{(1)} - \tilde{\sigma}^{-1} \cdot \tilde{\alpha} \cdot \boldsymbol{g}^{e(3)}, \tag{7.8}$$

and then, by substitution, two equations of the form (7.4), with

$$\tilde{\rho} = c_{11}^{e} \boldsymbol{I} + c_{13}^{e} \tilde{\alpha} \cdot \tilde{\sigma}^{-1} + c_{15}^{e} \tilde{\beta} \cdot \tilde{\sigma}^{-1}, \tag{7.9}$$

$$\tilde{\boldsymbol{b}} = c_{13}^{e} (\tilde{\alpha} \cdot \tilde{\sigma}^{-1} \cdot \tilde{\alpha} - \tilde{\kappa}^{e}) + c_{15}^{e} (\tilde{\beta} \cdot \tilde{\sigma}^{-1} \cdot \tilde{\alpha} - \tilde{\gamma}), \tag{7.10}$$

$$\tilde{\boldsymbol{b}}' = \tilde{\alpha} \cdot \tilde{\sigma}^{-1}, \tag{7.11}$$

$$\tilde{\kappa}'^{e} = \tilde{\kappa}^{e} - \tilde{\alpha} \cdot \tilde{\sigma}^{-1} \cdot \tilde{\alpha}. \tag{7.12}$$

We note that all the matrices $\tilde{\rho}$, $\tilde{\boldsymbol{b}}$, $\tilde{\boldsymbol{b}}'$, $\tilde{\kappa}'^{e}$ have the characteristic form (5.22) required by the isotropy in the plane perpendicular to $\boldsymbol{b}$. We now discuss these relations in some detail (in the reverse order, for convenience).

(a) *Braginskii's thermal conductivity*. The form of eq. (7.12) is independent of the truncation level. The three independent coefficients of the tensor $\tilde{\kappa}'^{e}$ are

$$\tilde{\kappa}'^{e}_{\parallel} = \tilde{\kappa}^{e}_{\parallel} - \frac{\tilde{\alpha}^2_{\parallel}}{\tilde{\sigma}_{\parallel}}, \qquad \tilde{\kappa}'^{e}_{\perp} = \tilde{\kappa}^{e}_{\perp} - \frac{1}{D_{\perp}}\left[\left(\tilde{\alpha}^2_{\perp} - \tilde{\alpha}^2_{\wedge}\right)\tilde{\sigma}_{\perp} + 2\tilde{\alpha}_{\perp}\tilde{\alpha}_{\wedge}\tilde{\sigma}_{\wedge}\right],$$

$$\tilde{\kappa}'^{e}_{\wedge} = \tilde{\kappa}^{e}_{\wedge} - \frac{1}{D_{\perp}}\left[2\tilde{\alpha}_{\perp}\tilde{\alpha}_{\wedge}\tilde{\sigma}_{\perp} + \left(\tilde{\alpha}^2_{\wedge} - \tilde{\alpha}^2_{\perp}\right)\tilde{\sigma}_{\wedge}\right], \tag{7.13}$$

where

$$D_{\perp} = \tilde{\sigma}^2_{\perp} + \tilde{\sigma}^2_{\wedge}. \tag{7.14}$$

These coefficients have a simple interpretation: they determine the *electron heat flux due to a temperature gradient when the electric current vanishes*,

$$\boldsymbol{q}^{e} = -\boldsymbol{\kappa}'^{e} \cdot \nabla T_{e}, \quad [\boldsymbol{j} = 0]. \tag{7.15}$$

The tensor $\boldsymbol{\kappa}'^{e}$ differs from $\boldsymbol{\kappa}^{e}$, which determines the heat flux when the effective field vanishes (e.g. when there is no electric field, no mass motion and no electron pressure gradient). The reason for this difference can be seen in eq. (7.8). When the current $\boldsymbol{h}^{(1)}$ vanishes, there appears an effective field $\boldsymbol{g}^{(1)}$ proportional to the temperature gradient: its effect must be added to the "normal" thermal conductivity in eq. (7.5).

Braginskii's parallel "zero-current thermal conductivity" is always *less* than the ordinary thermal conductivity,

$$\tilde{\kappa}'^{e}_{\parallel} \leqslant \tilde{\kappa}^{e}_{\parallel},$$

as can be seen from the first eq. (7.13). Its positive sign results from the thermodynamic inequality

$$\tilde{\kappa}^{e}_{\parallel}\tilde{\sigma}_{\parallel} - \tilde{\alpha}^2_{\parallel} \geqslant 0,$$

which will be derived in chapter 6. The sign of the differences $\tilde{\kappa}'^{e}_{\perp} - \tilde{\kappa}^{e}_{\perp}$, $\tilde{\kappa}'^{e}_{\wedge} - \tilde{\kappa}^{e}_{\wedge}$ is not imposed by thermodynamics. It turns out, from the numerical evaluations, that $\tilde{\kappa}'^{e}_{\wedge} - \tilde{\kappa}^{e}_{\wedge} \leqslant 0$; $\tilde{\kappa}'^{e}_{\perp} - \kappa^{e}_{\perp}$ is negative for small $|x_e| \equiv |\Omega_e|\tau_e$, but changes sign for larger values. We may also note that, for large $|x_e|$, $\tilde{\kappa}'^{e}_{\wedge} \approx \tilde{\kappa}^{e}_{\wedge}$. The transverse zero-current thermal conductivities are plotted in fig. 7.1.

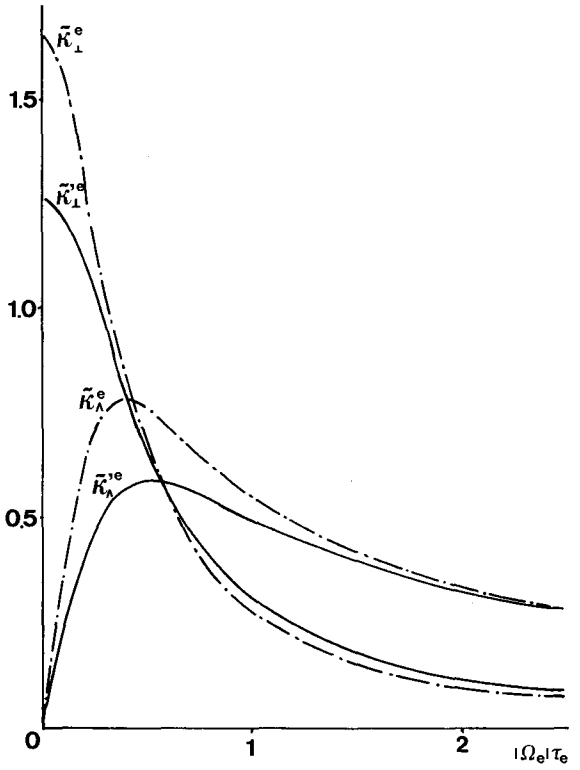

Fig. 7.1. Braginskii's zero-current electron thermal conductivities $\tilde{\kappa}_A'^e$ compared to the zero-field electron thermal conductivities $\tilde{\kappa}_A^e$.

Finally, we note that the coefficients $\tilde{\kappa}_\parallel'^e$, $\tilde{\kappa}_\perp'^e$, $\tilde{\kappa}_\wedge'^e$ calculated by using (7.13) with our 21M transport coefficients agree with those calculated by Braginskii's formulae within two decimal figures over the whole range of $|x_e|$.

(b) *Braginskii's thermoelectric coefficients*. Here, a quite interesting point appears. The coefficient $\tilde{\boldsymbol{b}}'$ is expressed in eq. (7.11) in a form independent of the approximation, whereas $\tilde{\boldsymbol{b}}$ is expressed in (7.10) in a form depending on the truncation level. Nevertheless, it can be shown by direct evaluation, using the expression of table 3.2,

$$\tilde{\boldsymbol{b}} = \tilde{\boldsymbol{b}}' = \tilde{\alpha} \cdot \tilde{\sigma}^{-1}. \tag{7.16}$$

This equality is an expression of the *Onsager principle*, whose validity is thus proved in Braginskii's formalism. It justifies the use of $(\tau_e \boldsymbol{Q}^{(1)}, \boldsymbol{h}^{(1)})$ and $(\boldsymbol{h}^{e(3)}, \boldsymbol{g}^{e(3)})$ as pairs of thermodynamically conjugate fluxes and forces.

(c) *Braginskii's resistivity tensor.* The tensor $\tilde{\rho}$ is defined by eq. (7.9) in an approximation-dependent form. We note that (7.10) together with (7.16) provides us with

$$c_{13}^{e}\tilde{\alpha}\cdot\tilde{\sigma}^{-1}+c_{15}^{e}\tilde{\beta}\cdot\tilde{\sigma}^{-1}=\left(\tilde{\alpha}\cdot\tilde{\sigma}^{-1}+c_{13}^{e}\tilde{\kappa}^{e}+c_{15}^{e}\tilde{\gamma}\right)\cdot\tilde{\alpha}^{-1}. \tag{7.17}$$

If we note, moreover,

$$\tilde{\alpha}\cdot\tilde{\sigma}^{-1}=\tilde{\sigma}^{-1}\cdot\tilde{\alpha},$$

we find that (7.9) can be transformed into

$$\tilde{\rho}=\tilde{\sigma}^{-1}+\tilde{\rho}', \tag{7.18}$$

where the tensor $\tilde{\rho}'$ is defined (in 21M) by

$$\tilde{\rho}'=\left(c_{11}^{e}\tilde{\alpha}+c_{13}^{e}\tilde{\kappa}^{e}+c_{15}^{e}\tilde{\gamma}\right)\cdot\tilde{\alpha}^{-1}. \tag{7.19}$$

When this tensor is worked out and the expression of the transport coefficients are taken from table 3.2 (an easy operation in 13M, but tedious for the transverse ones in 21M), the following surprising result is found: *the tensor* $\tilde{\rho}'$ *has vanishing diagonal elements* ($\tilde{\rho}'_{\parallel}=\tilde{\rho}'_{\perp}=0$), *but non-zero non-diagonal elements* ($\tilde{\rho}'_{\wedge}\neq 0$). This, in turn, implies the following statement: *The diagonal elements of Braginskii's tensor* $\tilde{\rho}$ *coincide with the diagonal elements of the inverse conductivity tensor* $\tilde{\sigma}^{-1}$; this is not the case for the off-diagonal elements,

$$\tilde{\rho}_{\parallel}=\left(\tilde{\sigma}^{-1}\right)_{\parallel},\qquad \tilde{\rho}_{\perp}=\left(\tilde{\sigma}^{-1}\right)_{\perp},\qquad \tilde{\rho}_{\wedge}\neq\left(\tilde{\sigma}^{-1}\right)_{\wedge}. \tag{7.20}$$

Thus, Braginskii's interpretation of the tensor $\tilde{\rho}$ as the *resistivity tensor* is – strangely enough – only "partially" valid. *.

In a very recent paper, Epperlein and Haines (1986) recalculated all the vector transport coefficients by solving *numerically* the kinetic equation for f_1 ($f\equiv f_0+f_1\cdot v$). They claim that the coefficients $\tilde{\rho}_{\wedge}$ and $\tilde{b}_{\perp}$ (in our notations) not only differ widely from Braginskii's values, but have a different asymptotic behaviour for $|x_e|\to\infty$. They support this point by an asymptotic analytical calculation which yields the behaviour $\tilde{\rho}_{\wedge}\sim|x_e|^{-2/3}$, $\tilde{b}_{\perp}\sim|x_e|^{-5/3}$. The

* One may note that Braginskii is – misleadingly – cautious. He gives general formulae (4.30), (4.34) for his coefficients $\alpha_{\parallel}$, $\alpha_{\perp}$, $\alpha_{\wedge}$ (proportional to our $\tilde{\rho}_{\parallel}$, $\tilde{\rho}_{\perp}$, $\tilde{\rho}_{\wedge}$), but *without giving them a name*! In his §2 ("Summary of results"), eq. (2.6), he gives *limiting expressions* of $\tilde{\rho}$ for very large $|x_e|$; in this limit, $\tilde{\rho}_{\wedge}\ll\tilde{\rho}_{\parallel}$, $\tilde{\rho}_{\perp}$ and the Hall coefficient is neglected. The resulting *approximately diagonal* tensor can then be identified with the *approximately diagonal* resistivity tensor!

validity of their complicated asymptotic expansion appears doubtful. Moreover, they do not show why all the *other* asymptotic transport coefficients have a "regular" behaviour. Their numerical results, which are hardly controllable and are based on empirical polynomial fits, contradict all the existing works, in particular the recent Kaneko 60-polynomial calculations. We are not ready to accept them without a further independent confirmation.

We have shown in section 5.6 that all the transport coefficients (in our formulation) have simple and universal limiting values, given by eq. (6.2) independently of the truncation level. There is, however, a small problem, precisely with *Braginskii's* coefficients $\tilde{\rho}_\wedge$ and $\tilde{b}_\perp$. As shown above, $\tilde{\rho}_\wedge$ does not have a *universal* expression, therefore its asymptotic value is not easily determined. As for $\tilde{b}_\perp$, it is given by

$$\tilde{b}_\perp = (\tilde{\alpha} \cdot \tilde{\sigma}^{-1})_\perp = \frac{\tilde{\alpha}_\perp \tilde{\sigma}_\perp + \tilde{\alpha}_\wedge \tilde{\sigma}_\wedge}{\tilde{\sigma}_\perp^2 + \tilde{\sigma}_\wedge^2} \sim \frac{x^{-2}x^{-2} + x^{-3}x^{-1}}{x^{-2}}.$$

The asymptotic behaviour of $\tilde{b}_\perp$ is defined by *both* terms in the numerator; $\tilde{\alpha}_\wedge$ behaves like x^{-3}, but with a non-universal coefficient (see section 5.6). Hence, $\tilde{b}_\perp$ *behaves like* x^{-2}, but its *coefficient* is not a simple, universal matrix element c^{e}_{ij}. On the contrary, it is easily checked that $\tilde{b}_\wedge$ does have a universal behaviour, $\tilde{b}^{\infty}_\wedge = c^{\mathrm{e}}_{13} x^{-1}$.

In conclusion, this discussion confirms the greater physical transparency of our formulation of the transport equations as compared to Braginskii's.

We now consider relations (7.20) in more detail. First, we note that, because of the peculiar form of the matrix $\tilde{\sigma}$, eq. (5.22) we have

$$\tilde{\rho}_\parallel = \frac{1}{\tilde{\sigma}_\parallel}. \tag{7.21}$$

The identification of the parallel resistivity as the inverse parallel conductivity is thus straightforward. But for the perpendicular resistivity one must be much more careful: $\tilde{\rho}_\perp$ is the perpendicular element of the inverse conductivity *matrix*

$$\tilde{\rho}_\perp = \frac{\tilde{\sigma}_\perp}{\tilde{\sigma}_\perp^2 + \tilde{\sigma}_\wedge^2}. \tag{7.22}$$

Here we note that *Braginskii makes a serious mistake* when he writes his formula (2.6) in a form implying

$$\left[\tilde{\rho}_\perp = \frac{1}{\tilde{\sigma}_\perp}\right]. \qquad \text{[Braginskii]} \tag{7.22'}$$

This expression is *approximately* valid only for $|x_e| \ll 1$, when $\tilde{\sigma}_\wedge \ll \tilde{\sigma}_\perp$, and certainly *not* in the opposite limit, for which Braginskii's result is quoted.

Actually, the behaviour of the perpendicular resistivity in the limit $|x_e| \gg 1$ is very interesting. As we know from (6.2), both $\tilde{\sigma}_\wedge$ and $\tilde{\sigma}_\perp$ tend to zero for large $|x_e|$ (hence, Braginskii's expression (7.22′) tends to infinity!). But $\tilde{\sigma}_\perp \sim x_e^{-2}$ and $\tilde{\sigma}_\wedge \sim x_e^{-1}$; hence, in this limit,

$$\tilde{\rho}_\perp^\infty = \frac{\tilde{\sigma}_\perp^\infty}{\tilde{\sigma}_\wedge^{\infty 2}} = c_{11}^e = 1. \tag{7.23}$$

Thus, the *perpendicular resistivity tends to a finite value as* $|x_e| \to \infty$, *although the perpendicular conductivity tends to zero.* This apparently paradoxical result dramatically stresses the difference between $(\tilde{\sigma}_\perp)^{-1}$ and $(\tilde{\sigma}^{-1})_\perp$. It also shows the essential role played by the Hall conductivity $\tilde{\sigma}_\wedge$ in the definition of the perpendicular resistivity. It is needless to add that the asymptotic result (7.23) is an *exact* result (see section 5.6).

If we now use the value of $\tilde{\sigma}_\parallel$ (for $Z = 1$) given in table 4.1, we obtain

$$\tilde{\rho}_\perp^\infty = 1.953\tilde{\rho}_\parallel. \tag{7.24}$$

This widely quoted (and sometimes misused!) relation, obtained by Braginskii, is quite correct; but the form given by this author [his eq. (2.8), as well as eq. (2.7)] is *incorrect*:

$$\tilde{\sigma}_\parallel = 1.96\tilde{\sigma}_\perp^\infty. \qquad \text{[Braginskii]}$$

We have plotted in fig. 7.2a the perpendicular resistivity, which demonstrates the smooth increase from $\tilde{\rho}_\perp(0)$ to $\tilde{\rho}_\perp^\infty$. In fig. 7.2b we plotted Braginskii's coefficient $\tilde{\rho}_\wedge$, compared to the Hall resistivity $(\tilde{\sigma}^{-1})_\wedge$. The difference is striking: the former tends to zero, whereas the latter tends linearly to infinity as $|x_e| \to \infty$.

(d) *Braginskii's other transport coefficients.* All the other transport equations, i.e. the ion heat flux and the ion and electron pressure tensors, are given by Braginskii in the same form as ours. His ion thermal conductivity and all his viscosities should therefore be directly comparable to ours. We have indeed checked the numerical values of all these coefficients with our 21M results and found an excellent agreement throughout the whole range of x_α. * This

* The reader should be warned about a peculiarity in Braginskii's definition of the relaxation times τ_e and τ_i [eqs. (2.5e) and (2.5i)]. His definition of τ_e is the same as our eq. (4.6.4), but in his definition of τ_i, the factor $(2\pi)^{1/2}$ [our eq. (4.6.10)] is replaced by $(\pi)^{1/2}$ and this is *not* a misprint! If our coefficients are compared to his, and this difference is overlooked, systematic differences by factors $(2)^{1/2}$ appear in all the ionic transport coefficients. We cannot understand his motivation for introducing such an artificial dissymmetry between ions and electrons!

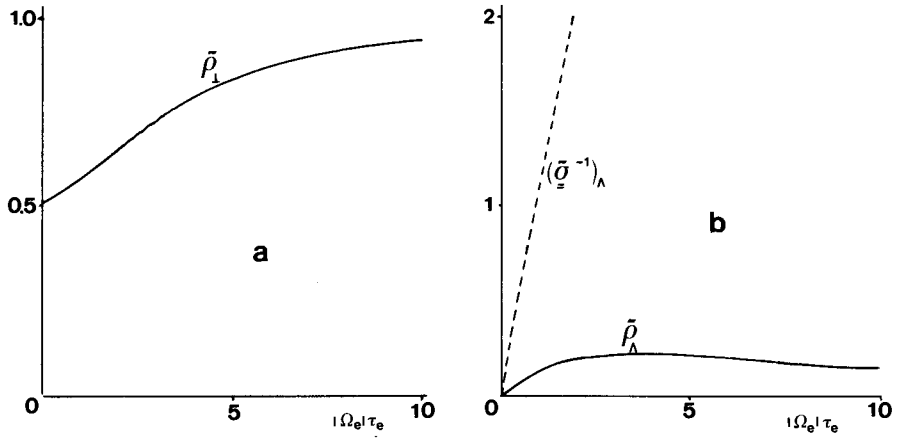

Fig. 7.2. (a) Braginskii's perpendicular electrical resistivity; (b) Braginskii's non-diagonal electrical resistivity, compared to the Hall resistivity.

agreement is the final test of the *equivalence between the Chapman–Enskog–Braginskii method and our moment method.* This agreement is so much more striking that the analytic forms are sometimes very different in the two cases (for instance, our expression for $\tilde{\kappa}^e_\perp$ is a ratio of polynomial of degree 4 to a polynomial of degree 6, whereas Braginskii's expression for $\tilde{\kappa}'^e_\perp$ is the ratio of a polynomial of degree 2 to a polynomial of degree 4). In spite of this, the difference between our 21M values and Braginskii's never exceeds 1%.

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6

Entropy and transport

6.1. *Entropy balance and H-theorem*

In chapter 3 (section 3.5) we discussed the properties of the entropy of a plasma by adopting a purely macroscopic point of view. Such an approach is limited by the assumptions necessary for grabbing a starting point. Here, we take up the problem again from a fundamental point of view and show how the macroscopic assumptions can be justified and how their limits of validity can be determined.

Even in this approach, our ambitions must be limited within certain bounds. To be specific, we assume the validity of a *kinetic equation* for the one-particle distribution functions $f^{\alpha}(\boldsymbol{v}; \boldsymbol{x}, t)$, and more particularly, the applicability of the Landau kinetic equation. The more general question of the existence and definition of the entropy for more general systems and more extended regimes will not be discussed here.

It is well known (Balescu 1975) that the entropy is definitely distinct from all the other hydrodynamical quantities defined in chapter 3. In particular, *the entropy cannot be defined as an average of a microscopic dynamical function*, like the energy, the pressure, etc. Its microscopic definition is somehow related to the integral over the whole phase space of the function $F \ln F$, where F is the N-particle distribution function. Thus, the entropy is a *collective quantity*, whose value is determined by the instantaneous state of the whole system.

In order to make this concept operational in a kinetic theory, and therefore in hydrodynamics, a definition is needed in terms of reduced distribution functions. This is a very complicated problem at the fully general level, because it involves a nonlinear functional of the correlation functions of arbitrary numbers of particles. (In contrast, the most general definition of the internal energy of a plasma, including the Coulomb interactions, involves only the one-particle and the two-particle distribution functions.)

However, for the weakly coupled plasmas studied in our book, a considerable simplification can be achieved. Indeed, it was shown in chapter 3 that these plasmas behave thermodynamically like ideal gases. Therefore, when we look for definitions of thermodynamical quantities, *the effect of the correlation functions can be neglected*. This statement is consistent with our definitions

(3.2.7) of the internal energy density, (3.2.18) of the pressure tensor and (3.2.21) of the heat flux. In all these quantities, the potential energy contributions, which are determined by the two-particle correlations, are neglected. One is then tempted to adopt for the entropy per particle the well-known, quite general definition of Boltzmann (1872), which, for a single component gas, is

$$s = -\frac{1}{N}\int \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{v} f(\boldsymbol{q}, \boldsymbol{v}, t)[\ln f(\boldsymbol{q}, \boldsymbol{v}, t) - 1] + \ln\frac{m^3}{h^3}$$

[Balescu (1975), eq. (7.3.12)]. (Here h is the Planck constant; we recall that the Boltzmann constant k_B is set equal to 1). This definition is, however, inconvenient for the study of spatially inhomogeneous systems, because it involves an integration over $\boldsymbol{q}$. The resulting entropy s is therefore solely a function of time. In transport theory, we need an entropy density defined *locally* at each point $\boldsymbol{x}$ in space and at each time t. We therefore adopt a slightly different definition for the *entropy density (per unit volume) of species* α, involving the *local* distribution functions (3.1.10),

$$n_\alpha(\boldsymbol{x}, t)s_\alpha(\boldsymbol{x}, t)$$

$$= -\int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)[\ln f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t) - 1] + n_\alpha(\boldsymbol{x}, t)\ln\frac{m_\alpha^3}{h^3}. \qquad (1.1)$$

The final justification for this definition will appear at the end of this section. A similar definition was adopted by de Groot and Mazur (1984), by Silin (1971), and by Misguich and Balescu (1984). Equivalently, eq. (1.1) can be written in the more compact form

$$n_\alpha(\boldsymbol{x}, t)s_\alpha(\boldsymbol{x}, t) = -\int \mathrm{d}\boldsymbol{v} f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)\ln\left(\frac{h^3}{\mathrm{e}m_\alpha^3} f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)\right), \qquad (1.2)$$

where e is the basis of natural logarithms, $\ln \mathrm{e} = 1$.

The total entropy density of the plasma may be defined by

$$n(\boldsymbol{x}, t)s(\boldsymbol{x}, t) = \sum_\alpha n_\alpha(\boldsymbol{x}, t)\, s_\alpha(\boldsymbol{x}, t).$$

It is, however, much more natural to study *separately* the two entropy densities per species, in the spirit of the two-fluid picture. This is especially advisable in a plasma whose electrons and ions do not have the same temperature.

We now derive the equation of evolution for the entropy density, by applying the usual method of chapter 3. Starting from (1.2), we find, using (**3**.3.1)–(**3**.3.3),

$$\partial_t(n_\alpha s_\alpha) = -\int d\boldsymbol{v}\left[\ln\left(\frac{h^3}{em_\alpha^3}f^\alpha\right)+1\right]\partial_t f^\alpha = \int d\boldsymbol{v}\left[\ln\left(\frac{h^3}{em_\alpha^3}f^\alpha\right)+1\right]$$
$$\times\left[-\boldsymbol{v}\cdot\nabla f^\alpha - \frac{e_\alpha}{m_\alpha}\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\wedge\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{v}}f^\alpha + \sum_\beta \mathscr{X}^{\alpha\beta}\right]$$
$$\equiv T_1 + T_2 + T_3. \tag{1.3}$$

The first term between brackets yields, after some elementary manipulations similar to those of section **3**.4,

$$T_1 = \nabla_r\int d\boldsymbol{v}\left(\ln\frac{h^3}{em_\alpha^3}f^\alpha\right)(v_r - u_r^\alpha)f^\alpha - \nabla_r(u_r^\alpha n_\alpha s_\alpha), \tag{1.4}$$

where $\boldsymbol{u}^\alpha$ is, as usual, the average velocity of species α.

The second term is easily shown to vanish identically. For the third term, we introduce the notation

$$\sigma^\alpha = \sum_\beta \sigma^{\alpha\beta}, \tag{1.5}$$

with

$$\sigma^{\alpha\beta}(\boldsymbol{x}, t) = -\int d\boldsymbol{v}[\ln f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)]\,\mathscr{X}^{\alpha\beta}. \tag{1.6}$$

[We note that the constant terms $[\ln(h^3/em_\alpha^3)+1]$ give a vanishing contribution to the collisional term, because of the number conservation property (**2**.7.4).] We also define

$$\boldsymbol{J}_S^\alpha(\boldsymbol{x}, t) = -\int d\boldsymbol{v}\,\ln\left(\frac{h^3}{em_\alpha^3}f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t)\right)[\boldsymbol{v}-\boldsymbol{u}^\alpha(\boldsymbol{x}, t)]\,f^\alpha(\boldsymbol{v}; \boldsymbol{x}, t). \tag{1.7}$$

The equation of evolution (1.3) can therefore be written in the form of an *entropy balance equation*,

$$\partial_t(n_\alpha s_\alpha) = -\nabla\cdot(n_\alpha s_\alpha \boldsymbol{u}^\alpha + \boldsymbol{J}_S^\alpha) + \sigma^\alpha. \tag{1.8}$$

We clearly recognize the convective entropy flux $n_\alpha s_\alpha \boldsymbol{u}^\alpha$, the (conductive) *entropy flux* $\boldsymbol{J}_s^\alpha$ and the *entropy source* σ^α.

Let us emphasize once more that we obtained two separate balance equations for the electron and the ion entropy densities. This two-fluid picture is different from the "traditional" description (de Groot and Mazur 1984), in which a single entropy density characterizes a mixture of gases,

$$\partial_t \rho s = -\nabla\cdot(\rho s \boldsymbol{u} + \boldsymbol{J}_s) + \sigma, \qquad \text{[de Groot and Mazur]}$$

where $\boldsymbol{u}$ is the centre-of-mass velocity. This type of description is unsuitable when the plasma is described by two temperatures and two heat fluxes.

We now come to the most important point. In order to evaluate the collisional entropy source, we use the Landau collision term in its symmetric form (2.7.7) and perform an integration by parts,

$$\begin{aligned}\sigma^{\alpha\beta} = &-\frac{A_{\alpha\beta}}{2}\int \mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{v}_2\,\ln\left(\frac{h^6}{e^2 m_\alpha^3 m_\beta^3} f^\alpha(\boldsymbol{v}_1;\,\boldsymbol{x},\,t) f^\beta(\boldsymbol{v}_2;\,\boldsymbol{x},\,t)\right)\\ &\times\left(m_\alpha^{-1}\,\partial_{1r} - m_\beta^{-1}\,\partial_{2r}\right) G_{rs}(\boldsymbol{g})\left(m_\alpha^{-1}\,\partial_{1s} - m_\beta^{-1}\,\partial_{2s}\right)\\ &\times f^\alpha(\boldsymbol{v}_1;\,\boldsymbol{x},\,t) f^\beta(\boldsymbol{v}_2;\,\boldsymbol{x},\,t)\\ = &\tfrac{1}{2}A_{\alpha\beta}\int \mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{v}_2\left[f^\alpha(\boldsymbol{v}_1;\,\boldsymbol{x},\,t) f^\beta(\boldsymbol{v}_2;\,\boldsymbol{x},\,t)\right]^{-1} W_r G_{rs} W_s,\end{aligned} \tag{1.9}$$

where we introduced the abbreviation

$$W_r \equiv \left(m_\alpha^{-1}\,\partial_{1r} - m_\beta^{-1}\,\partial_{2r}\right) f^\alpha(\boldsymbol{v}_1;\,\boldsymbol{x},\,t) f^\beta(\boldsymbol{v}_2;\,\boldsymbol{x},\,t). \tag{1.10}$$

If $\boldsymbol{A}$ is *any* vector, making an angle θ with the relative velocity vector $\boldsymbol{g}$, the following property is easily derived from definition (3.3.9) of the Landau tensor:

$$g^3 A_r G_{rs} A_s = A_r\left(g^2\delta_{rs} - g_r g_s\right) A_s = A^2 g^2 - (\boldsymbol{A}\cdot\boldsymbol{g})^2 = A^2 g^2 \sin^2\theta \geqslant 0. \tag{1.11}$$

As the distribution functions f^α are clearly non-negative quantities, we conclude from (1.9) and (1.11)

$$\sigma^{\alpha\beta} \geqslant 0. \tag{1.12}$$

The inequality is valid for *each pair of indices*, e–e, e–i, i–e, i–i. Equation (1.12) may be called the *H-theorem for a plasma*. It follows from eq. (1.8) that, in each point x and at all times, the entropy of each species can never decrease as a result of the collisions. For this reason, the entropy source $\sigma^{\alpha\beta}$ will be more adequately called the *entropy production*.

The H-theorem is the final justification of the choice (1.1) or (1.2) for the local entropy density. It constitutes a *proof* of the second law of thermodynamics in the case of a weakly coupled plasma.

It must be stressed, however, that *the result* (1.12) *is stronger than the mere statement of the second law*. Indeed, the latter only requires the *total* entropy production to be non-negative,

$$\sigma \equiv \sum_\alpha \sum_\beta \sigma^{\alpha\beta} \geqslant 0, \tag{1.13}$$

whereas the kinetic theory tells us that *each term in this sum is separately non-negative*. In other words, every collision process is a very effective "entropy producer". It is impossible to realize a situation where, for instance, $\sigma^{ee} > 0$, $\sigma^{ei} < 0$, but $\sigma^{ee} + \sigma^{ei} > 0$. Such a situation would be acceptable by the second law, but is excluded by kinetic theory.

We finally note that (1.12) identifies the *collisions* as the *only source of irreversibility* or dissipation. This statement is valid whenever the plasma is "quiescent". According to our definition (section 3.1), the statistical state of a quiescent plasma is completely described by the one-particle distribution functions f^α. If the plasma becomes unstable and turbulent, this description becomes incomplete and one must take into account correlations and collective phenomena. One then finds additional sources of dissipation, leading to the so-called *anomalous transport*. This important problem will however not be addressed in the present volume.

6.2. *Entropy and Hermitian moments. The kinetic form of the entropy production*

All the results of section 6.1 are completely general, i.e. are independent of any assumption about the state of the plasma.

We now wish to specialize these results to the plasma regime considered in classical transport theory. The state is then close to a local plasma equilibrium and the distribution functions have the form (4.3.8), completed by expansions (4.3.19)–(4.3.22). In this case, the various quantities related to the entropy can be calculated explicitly.

Consider first the entropy density. Substituting (4.3.8) into (1.2) and using the dimensionless variable c, eq. (4.3.4), we find

$$n_\alpha s_\alpha = -n_\alpha \int \mathrm{d}c\ \phi^0(c)[1+\chi^\alpha(c)]$$

$$\times \ln\left[\frac{h^3}{e m_\alpha^3} n_\alpha \left(\frac{m_\alpha}{T_\alpha}\right)^{3/2} \phi^0(c)[1+\chi^\alpha(c)]\right]. \tag{2.1}$$

The entropy per particle can therefore be written in the form

$$s_\alpha = s_\alpha^{[0]} + s_\alpha^{[1]} + \cdots, \tag{2.2}$$

where $s_\alpha^{[n]}$ is a functional of order n in the deviation χ^α. The zeroth order term is (recalling eq. 4.3.6)

$$s_\alpha^{[0]} = -\ln\left(\frac{h^3 n_\alpha}{(2\pi m_\alpha T_\alpha)^{3/2}}\right) + 1 + \tfrac{1}{2}\int \mathrm{d}c\ \phi^0(c)\ c^2,$$

$$s_\alpha^{[0]} = -\ln\left(\frac{h^3}{(2\pi m_\alpha)^{3/2}} \frac{n_\alpha}{T_\alpha^{3/2}}\right) + \frac{5}{2}. \tag{2.3}$$

This is precisely the *local equilibrium entropy per particle*, as given in eq. (3.5.8). As for the first-order term, we find [using $\ln(1+\chi) \approx \chi$]

$$s_\alpha^{[1]} = -\int \mathrm{d}c\ \phi^0 \chi^\alpha \left(\ln \frac{h^3 n_\alpha}{(2\pi m_\alpha T_\alpha)^{3/2}} - \tfrac{1}{2}c^2\right) = 0, \tag{2.4}$$

the vanishing being due to constraints (4.3.9). We have thus justified the *quasi-thermodynamical definition of the entropy density* used in section 3.5, with the additional result that the deviation from $s_\alpha^{[0]}$ is at least quadratic in χ^α. This important point was established (for neutral gases) by Prigogine (1949) (see also de Groot and Mazur 1984, ch. IX, §6).

Next, we consider the various terms in the entropy balance equation (1.8), starting with the (conductive) *entropy flux*. A short calculation shows that, through linear order in χ^α,

$$J^\alpha_{Sr} = -n_\alpha \int \mathrm{d}\boldsymbol{c}\ \phi^0(1+\chi^\alpha) c_r \ln\left(\frac{h^3}{e m_\alpha^3} n_\alpha \frac{m_\alpha}{T_\alpha^{3/2}} \phi^0(1+\chi^\alpha)\right)$$

$$= \tfrac{1}{2} n_\alpha \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \int \mathrm{d}\boldsymbol{c}\ \phi^0 \chi^\alpha c^2 c_r$$

$$= \sqrt{\tfrac{5}{2}}\, n_\alpha \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \int \mathrm{d}\boldsymbol{c}\ \phi^0 \chi^\alpha H_r^{(3)}(\boldsymbol{c}).$$

[In this calculation, constraints (4.3.9) are repeatedly used]. Thus, the entropy flux is proportional to the Hermitian moment $h_r^{\alpha(3)}$, i.e. to the heat flux. Using the definition of table 3.2.1 ("Fluxes") we get

$$J^\alpha_{Sr} = \sqrt{\tfrac{5}{2}}\, n_\alpha \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} h_r^{\alpha(3)} = \frac{1}{T_\alpha} q_r^\alpha. \tag{2.5}$$

We thus recover, to leading order, the non-equilibrium thermodynamics result (3.5.15).

We now go over to the most important part of our investigation, i.e. the evaluation of the *entropy production*. The explicit calculation is straightforward, but rather tedious. We therefore only outline the steps, and give the final result.

– For the electron–electron and the ion–ion entropy production (σ^{ee}, σ^{ii}) we evaluate eq. (1.9), combined with (4.3.8).

– The electron–ion entropy production is evaluated by using the "Lorentz approximation" (2.8.1), (2.8.11) of the collision operator. A calculation similar to the one leading to (1.9) yields

$$\sigma^{ei} = \tfrac{1}{2} A_{ei} m_{ei}^{-2} \int \mathrm{d}\boldsymbol{v}\ \frac{1}{f^e(\boldsymbol{v};\ \boldsymbol{x},\ t)} V_r \mathscr{G}_{rs} V_s, \tag{2.6}$$

where

$$V_r = \frac{\partial f^e(\boldsymbol{v};\ \boldsymbol{x},\ t)}{\partial v_r},$$

$$\mathscr{G}_{rs} = n_i \left(G_{rs}(\boldsymbol{v}) + u_n^{ei} \frac{\partial}{\partial v_n} G_{rs}(\boldsymbol{v}) + \tfrac{1}{2} \alpha_{mn} \frac{\partial^2}{\partial v_m\, \partial v_n} G_{rs}(\boldsymbol{v}) \right). \tag{2.7}$$

(It may be checked that the positive sign of the entropy production is not spoiled by the Lorentz approximation.) Equation (2.6) is also combined with (**4**.3.8).

– The ion–electron entropy production gives contributions of relative order $\mu = m_e/m_i$ and is therefore neglected.

– It is found that the leading order to the entropy production is *quadratic* in the deviations χ^α, hence quadratic in the Hermitian moments $h^{\alpha(n)}_{\cdots}$. We only retain these quadratic terms, neglecting higher-order non-linearities.

– A rather long calculation, of the same type as those explained in section **4**.6, leads to a remarkably simple result,

$$\sigma^e \equiv \sigma^{ee} + \sigma^{ei} = \frac{n_e}{\tau_e}\left[\sum_{p=0}^{N}\sum_{q=0}^{N} c^e_{2p+1,2q+1} h_r^{e(2p+1)} h_r^{e(2q+1)} + \sum_{p=1}^{N}\sum_{q=1}^{N} c^e_{2p,2q} h_{rs}^{e(2p)} h_{rs}^{e(2q)} + \cdots\right], \tag{2.8}$$

$$\sigma^i \equiv \sigma^{ii} = \frac{n_i}{\tau_i}\left[\sum_{p=1}^{N}\sum_{q=1}^{N} c^i_{2p+1,2q+1} h_r^{i(2p+1)} h_r^{i(2q+1)} + \sum_{p=1}^{N}\sum_{q=1}^{N} c^i_{2p,2q} h_{rs}^{i(2p)} h_{rs}^{i(2q)} + \cdots\right], \tag{2.9}$$

where N denotes the level of truncation (e.g. $N = 2$ for 21M). We recall the convention introduced in table **4**.6.2 and used in all the compact formulae of the present chapter,

$$h_r^{e(1)} \equiv h_r^{(1)}.$$

The coefficients c^e_{mn}, c^i_{mn} are precisely the Hermitian matrix elements of the collision term, i.e. the *same* numbers as those entering the transport coefficients, listed in tables **4**.6.3 and **5**.3.1. Both eqs. (2.8) and (2.9) can also be grouped in the single compact form

$$\sigma^\alpha = \frac{n_\alpha}{\tau_\alpha}\left[\sum_p\sum_q c^\alpha_{2p+1,2q+1} h_r^{\alpha(2p+1)} h_r^{\alpha(2q+1)} + \sum_p\sum_q c^\alpha_{2p,2q} h_{rs}^{\alpha(2p)} h_{rs}^{\alpha(2q)} + \cdots\right]. \tag{2.10}$$

We now comment on this result.

To leading order, *the entropy production appears as a quadratic form in the Hermitian moments of all orders and of all tensorial characters*. It therefore involves an infinite number of variables. In practice, each of this expression is truncated at an arbitrarily given level N.

Vectors and traceless tensors do not couple to each other: we thus find separate, independent contributions for each type of tensor moments.

The ellipsis $(\cdots)$ in eqs. (2.8)–(2.10) stands for the contribution of the scalar Hermitian moments, as well as for all the anisotropies of third and higher order. In the next section, a considerable simplification of this result will be achieved.

The most important fact to be stressed is that *the expression of the entropy production as a quadratic form in the Hermitian moments is an intrinsic one: its coefficients c^{α}_{mn} are solely determined by the properties of the collision term, which is, indeed, the real source of the irreversibility*. Moreover, the rate of entropy production is determined by the *collisional relaxation times* τ_{α}. In particular, the external electric or magnetic fields do not enter this expression at all.

The only assumption underlying (2.10) is the validity of eq. (4.3.8), expressing that the plasma is close to the local equilibrium state. Equation (2.10) will be called the *kinetic form of the entropy production*.

Consider now the *sign* of the entropy production. The (strong) second law of thermodynamics requires that both σ^{e} and σ^{i} be *positive definite quadratic forms*. Moreover, because of the mutual independence of vectors and tensors, we conclude that *each one* of the four partial quadratic forms of eq. (2.8), (2.9) is separately positive definite. This implies the existence of a set of relations between the coefficients c^{α}_{mn}. These can be found in any textbook on algebra (e.g. Smirnov 1970, Kurosh 1971, Korn and Korn 1968). It is easily checked, by using the values of table 4.6.3, that all these conditions are indeed verified at each truncation level (13M, 21M,...). We shall only quote here a few consequences of the positivity criteria.

- The diagonal coefficients c^{α}_{mm} are positive for every α and every m,

$$c^{\alpha}_{mm} > 0. \tag{2.11}$$

- The following second-order determinants are positive,

$$D^{\alpha}_{mn} \equiv c^{\alpha}_{mm} c^{\alpha}_{nn} - c^{\alpha 2}_{mn} > 0. \tag{2.12}$$

- The full nth order determinant (for any truncation level) is positive,

$$\| c^{\alpha}_{mn} \| > 0. \tag{2.13}$$

[In eqs. (2.12), (2.13), m and n must be both odd or both even.]

6.3. *The thermodynamic form of the entropy production*

The entropy production will be transformed into various alternative forms, which will bring out additional aspects of this important quantity.

We first recall the expressions (**4**.6.34) of the *generalized frictions*. It is then obvious that the entropy production can be expressed in the simple form

$$\sigma^{\alpha} = -n_{\alpha}\Big(\sum_{p} h_{r}^{\alpha(2p+1)} Q_{r}^{\alpha(2p+1)} + \sum_{p} h_{rs}^{\alpha(2p)} Q_{rs}^{\alpha(2p)}\Big). \tag{3.1}$$

Thus, the entropy production can be written as a *bilinear form in the Hermitian moments and the corresponding generalized frictions*.

This form suggests a pictorial analogy (which should not be taken too seriously!). The rate at which work is done by a force $\boldsymbol{F}$ on a particle moving with velocity $\boldsymbol{v}$ is $\boldsymbol{F}\cdot\boldsymbol{v}$, i.e. the product of the force by the "flux" $\boldsymbol{v}$. In (3.1), the entropy production is similarly expressed in terms of products of friction "forces" and "fluxes". This analogy suggests the relation between entropy production and energy dissipation.

We now transform the entropy production by introducing explicitly the *thermodynamic forces*, i.e. in dimensionless form, the source term $g_{\ldots}^{\alpha(m)}$. In order to do so, *we use the linearized moment equations* (**5**.1.7)–(**5**.1.10) (with $\partial_t h_{\ldots}^{\alpha(m)} = 0$),

$$\begin{aligned}
&-Q_{r}^{\alpha(2p+1)} = \tau_{\alpha}^{-1} g_{r}^{\alpha(2p+1)} + \epsilon_{rmn}\Omega_{\alpha} h_{m}^{\alpha(2p+1)} b_{n},\\
&-Q_{rs}^{\alpha(2p)} = \tau_{\alpha}^{-1} g_{rs}^{\alpha(2p)} + \big(\epsilon_{rmn} h_{sm}^{\alpha(2p)} + \epsilon_{smn} h_{rm}^{\alpha(2p)}\big)\Omega_{\alpha} b_{n},\\
&\quad p = 0, 1, 2, \ldots.
\end{aligned} \tag{3.2}$$

In order to calculate the entropy production by (3.1), we contract all the terms in the first equation with $h_{r}^{\alpha(2p+1)}$ and those of the second equation with $h_{rs}^{\alpha(2p)}$, and sum all the equations over p. We then note

$$\epsilon_{rmn} h_{r}^{\alpha(2p+1)} h_{m}^{\alpha(2p+1)} b_{n} \equiv 0,$$

$$\big(\epsilon_{rmn} h_{sm}^{\alpha(2p)} + \epsilon_{smn} h_{rm}^{\alpha(2p)}\big) h_{rs}^{\alpha(2p)} b_{n} \equiv 0.$$

Thus, *the magnetic terms drop out from the entropy production* and we are left with

$$\sigma^{\alpha} = \frac{n_{\alpha}}{\tau_{\alpha}} \sum_{p} \big(h_{r}^{\alpha(2p+1)} g_{r}^{\alpha(2p+1)} + h_{rs}^{\alpha(2p)} g_{rs}^{\alpha(2p)}\big). \tag{3.3}$$

This is not the end of the story! We recall that *the source terms $g_{\dots}^{\alpha(m)}$ are identically zero for all the non-privileged Hermitian moments*. Hence, the infinite sums over p in (3.3) are cut down to a *finite number of terms*, corresponding only to the *privileged Hermitian moments*

$$\sigma^{\mathrm{e}} = \frac{n_{\mathrm{e}}}{\tau_{\mathrm{e}}}\left(h_r^{(1)}g_r^{(1)} + h_r^{\mathrm{e}(3)}g_r^{\mathrm{e}(3)} + h_{rs}^{\mathrm{e}(2)}g_{rs}^{\mathrm{e}(2)}\right),$$

$$\sigma^{\mathrm{i}} = \frac{n_{\mathrm{i}}}{\tau_{\mathrm{i}}}\left(h_r^{\mathrm{i}(3)}g_r^{\mathrm{i}(3)} + h_{rs}^{\mathrm{i}(2)}g_{rs}^{\mathrm{i}(2)}\right). \tag{3.4}$$

These relations do not involve any approximation: they are, in particular, independent of the level of truncation. It must be stressed that this enormous simplification is possible because of our use of *irreducible Hermitian moments* for the description of the non-equilibrium plasma state. Only in this formulation can one make a distinction between privileged moments and non-privileged moments, whose source terms are identically zero.

It is now easily checked that eqs. (3.4), (3.5) are precisely equivalent to expressions (**3**.5.17) obtained from macroscopic thermodynamics. Indeed, from (3.2), (**4**.4.4) and (**4**.5.7) we find

$$\frac{n_{\mathrm{e}}}{\tau_{\mathrm{e}}}h_r^{(1)}g_r^{(1)} = -n_{\mathrm{e}}h_r^{(1)}Q_r^{(1)} = \frac{1}{T_{\mathrm{e}}}\boldsymbol{j}\cdot\boldsymbol{R}^{\mathrm{ei}}. \tag{3.5}$$

Also, from definitions (**5**.1.13)–(**5**.1.16) and table **3**.2.1 ("Fluxes") we find

$$\frac{n_\alpha}{\tau_\alpha}h_r^{\alpha(3)}g_r^{\alpha(3)} = -\frac{1}{T_\alpha^2}\boldsymbol{q}^\alpha\cdot\nabla T_\alpha,$$

$$\frac{n_\alpha}{\tau_\alpha}h_{rs}^{\alpha(2)}g_{rs}^{\alpha(2)} = -\frac{1}{T_\alpha}\Pi^\alpha : \nabla\boldsymbol{u}^\alpha. \tag{3.6}$$

Equation (3.4) may therefore be called the *thermodynamic form of the entropy production*.

It may be stressed that the derivation of eq. (3.4) (from kinetic theory) is completely independent of the thermodynamic derivation of section **3**.5. In particular, *no use was made of the hydrodynamic balance equations* (i.e. $\partial_t n_\alpha$, $\partial_t T_\alpha$).

We note the following important corollary of the previous derivation. Let $h_{\dots}^{\alpha(p)}$ be any *non-privileged* Hermitian moment. Our argument implies

$$\sum_q c_{pq}^\alpha\, h_{\dots}^{\alpha(p)}\, h_{\dots}^{\alpha(q)} = 0. \tag{3.7}$$

As a result, the *kinetic* form of the entropy production reduces considerably to

$$\sigma^{\mathrm{e}} = \frac{n_{\mathrm{e}}}{\tau_{\mathrm{e}}}\left[\sum_{q=0}^{N}\left(c^{\mathrm{e}}_{1,2q+1}h_r^{(1)} + c^{\mathrm{e}}_{3,2q+1}h_r^{\mathrm{e}(3)}\right)h_r^{\mathrm{e}(2q+1)} + \sum_{q=0}^{N} c^{\mathrm{e}}_{2,2q}h_{rs}^{\mathrm{e}(2)}h_{rs}^{\mathrm{e}(2q)}\right],$$

$$\sigma^{\mathrm{i}} = \frac{n_{\mathrm{i}}}{\tau_{\mathrm{i}}}\sum_{q=0}^{N}\left(c^{\mathrm{i}}_{3,2q+1}h_r^{\mathrm{i}(3)}h_r^{\mathrm{i}(2q+1)} + c^{\mathrm{i}}_{2,2q}h_{rs}^{\mathrm{i}(2)}h_{rs}^{\mathrm{i}(2q)}\right). \tag{3.8}$$

Thus, although there is still an infinite number of moments contributing to (3.8), these are only the vectorial and second-rank tensorial moments. *All the other anisotropies drop out from this expression*, and the ellipsis in (2.8)–(2.10) can be suppressed.

We now return to the thermodynamic form (3.4) of the entropy production. It is a *bilinear form in the thermodynamic fluxes* $h_{\cdots}^{\alpha(p)}$ *and their conjugate thermodynamic forces* $g_{\cdots}^{\alpha(p)}$. We now see that *only the privileged Hermitian moments can properly be called thermodynamic fluxes*: they are the only ones contributing to the thermodynamic form of the entropy production.

Another remarkable fact about eq. (3.4) is that *the coefficients in the bilinear form are independent of any characteristic microscopic parameter*. Indeed, the combination

$$\tau_\alpha^{-1} g_{\cdots}^{\alpha(p)}$$

is independent of τ_α (see eqs. 3.5 and 3.6). As a result, the form (3.4) is independent of any specific collision model [unlike the kinetic form (3.8)]. This constitutes its generality. On the other hand, it can only be evaluated consistently when the relation between the fluxes and the forces is known. Thus, the evaluation of the entropy production requires a transport theory.

6.4. *The transport form of the entropy production*

The combination of entropy production and transport theory is obtained by introducing explicitly the *transport equations* which relate the thermodynamic fluxes to the thermodynamic forces and were derived in chapter 5. We remain, of course, in the linear domain of transport theory, the only regime discussed

in this volume. The dimensionless transport equations can be written in the general form

$$h_r^{e(2p+1)} = \sum_{q=0,1} \tilde{L}_{rm}^{e(2p+1,2q+1)} g_m^{e(2q+1)}, \quad p = 0, 1,$$

$$h_r^{i(3)} = \tilde{L}_{rm}^{i(33)} g_r^{i(3)} m,$$

$$h_{rs}^{\alpha(2)} = \tilde{L}_{rsmn}^{\alpha(22)} g_{mn}^{\alpha(2)}, \tag{4.1}$$

where $\tilde{L}_{\ldots}^{\alpha(pq)}$ denotes the various dimensionless transport coefficients. When these equations are combined with the thermodynamic form of the entropy production (3.4), the latter quantity becomes a *quadratic form in the thermodynamic forces* $g_{\ldots}^{\alpha(p)}$, which is written explicitly as

$$\sigma^e = \frac{n_e}{\tau_e}\Big[\tilde{\sigma}_{rs} g_r^{(1)} g_s^{(1)} + \tilde{\alpha}_{rs}\big(g_r^{(1)} g_s^{e(3)} + g_r^{e(3)} g_s^{(1)}\big) + \tilde{\kappa}_{rs}^e g_r^{e(3)} g_s^{e(3)} + \tilde{\eta}_{rsmn}^e g_{rs}^{e(2)} g_{mn}^{e(2)}\Big],$$

$$\sigma^i = \frac{n_i}{\tau_i}\big(\tilde{\kappa}_{rs}^i g_r^{i(3)} g_s^{i(3)} + \tilde{\eta}_{rsmn}^i g_{rs}^{i(2)} g_{mn}^{i(2)}\big). \tag{4.2}$$

This will be called the *transport form of the entropy production*. It is the most explicit form, because it determines the entropy production in terms of the external constraints driving the plasma, i.e. the electric and magnetic fields and the gradients of temperature, pressure and velocity. In contrast to the "kinetic form", the transport form of the entropy production involves no *explicit* truncation level: only the privileged moments appear in (4.2). However, it must be kept in mind that the truncation level is implicitly contained i the values of the transport coefficients which, in practice, are evaluated in a given approximation.

When there is *no magnetic field* ($B = 0$), we know that the transport coefficients are scalars. The entropy production then reduces to

$$\sigma^e = \frac{n_e}{\tau_e}\big(\tilde{\sigma}_\parallel \boldsymbol{g}^{(1)} \cdot \boldsymbol{g}^{(1)} + 2\tilde{\alpha}_\parallel \boldsymbol{g}^{(1)} \cdot \boldsymbol{g}^{e(3)} + \tilde{\kappa}_\parallel^e \boldsymbol{g}^{e(3)} \cdot \boldsymbol{g}^{e(3)} + \tilde{\eta}_\parallel^e \boldsymbol{g}^{e(2)} : \boldsymbol{g}^{e(2)}\big),$$

$$\sigma^i = \frac{n_i}{\tau_i}\big(\tilde{\kappa}_\parallel^i \boldsymbol{g}^{i(3)} \cdot \boldsymbol{g}^{i(3)} + \tilde{\eta}_\parallel^i \boldsymbol{g}^{i(2)} : \boldsymbol{g}^{i(2)}\big), \qquad [B = 0]. \tag{4.3}$$

It is also interesting to write the right-hand sides in terms of the dimensional forces and transport coefficients, using eqs. (5.1.12)–(5.1.16), (5.3.11)–(5.3.13) and (5.3.21),

$$T_e\sigma^e = \sigma_{\parallel}\,\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{E}} - 2\alpha_{\parallel}\,\hat{\boldsymbol{E}}\cdot\nabla T_e + \kappa_{\parallel}^e\,\frac{1}{T_e}|\nabla T_e|^2 + \eta_{\parallel}^e\,\boldsymbol{\nu}:\boldsymbol{\nu},$$

$$T_i\sigma^i = \kappa_{\parallel}^i\,\frac{1}{T_i}|\nabla T_i|^2 + \eta_{\parallel}^i\,\boldsymbol{\nu}:\boldsymbol{\nu}, \qquad [B=0] \tag{4.4}$$

where the effective electric field $\hat{\boldsymbol{E}}$ was defined in (5.3.10) and the symmetrized velocity gradient $\boldsymbol{\nu}$ was defined in (5.3.19).

Thus, each transport coefficient contributes a term to the entropy production. The thermoelectric coefficient $\alpha_{\parallel}$ introduces a cross-term into σ^e, all other (diagonal) coefficients are associated with square terms. All these transport coefficients are thus connected to *dissipative (irreversible) processes* in the plasma.

Consider now the general case of *transport in the presence of a magnetic field*. The transport coefficients are now tensors. In the transport form of the entropy production, each of these tensors is contracted with two (vector or tensor) forces. The peculiar forms (5.5.30), (5.5.48) of the transport tensors must now be taken into account; these result from the isotropy in the plane perpendicular to the magnetic field. Each of the tensors can be decomposed into a symmetric and an antisymmetric part. It is clear that *none of the antisymmetric parts of the transport tensors can contribute to the entropy production*, which is a quadratic form.

Indeed, if $\boldsymbol{A}$ is any antisymmetric second rank tensor and $\boldsymbol{g}$ is any vector,

$$\boldsymbol{g}\cdot\boldsymbol{A}\cdot\boldsymbol{g} = 0.$$

Similarly, if $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{\nu}$ are arbitrary second rank tensors,

$$\boldsymbol{\nu}:(\boldsymbol{BC}-\boldsymbol{CB}):\boldsymbol{\nu} = 0.$$

It then follows from (5.5.31), (5.5.46), (5.5.47) that *all the non-diagonal transport coefficients* $\tilde{L}_{\wedge}$, *as well as the odd viscosities* $\tilde{\eta}_1^{\alpha}$, $\tilde{\eta}_3^{\alpha}$, *drop out from the entropy production. The corresponding fluxes are purely reversible, non-dissipative fluxes.*

We introduce the notations

$$g_{\parallel} = \boldsymbol{b} \cdot \boldsymbol{g}, \qquad \boldsymbol{g}_{\perp} = \boldsymbol{S}_4 \cdot \boldsymbol{g} = \boldsymbol{b} \wedge (\boldsymbol{g} \wedge \boldsymbol{b}),$$

$$\boldsymbol{\nu}_{\parallel} = \left(\boldsymbol{S}_0 \boldsymbol{S}_0 + \tfrac{1}{2} \boldsymbol{S}_4 \boldsymbol{S}_4\right) : \boldsymbol{\nu},$$

$$\boldsymbol{\nu}_2 = \tfrac{1}{2}\left(\boldsymbol{S}_3 \boldsymbol{S}_3 + \boldsymbol{S}_5 \boldsymbol{S}_5\right) : \boldsymbol{\nu},$$

$$\boldsymbol{\nu}_4 = \tfrac{1}{2}\left(\boldsymbol{S}_1 \boldsymbol{S}_1 + \boldsymbol{S}_2 \boldsymbol{S}_2\right) : \boldsymbol{\nu}. \tag{4.5}$$

The entropy production is then written in the form

$$\begin{aligned} \frac{\tau_e}{n_e} \sigma^e = {} & \tilde{\sigma}_{\parallel} g_{\parallel}^{(1)} g_{\parallel}^{(1)} + 2\tilde{\alpha}_{\parallel} g_{\parallel}^{(1)} g_{\parallel}^{e(3)} + \tilde{\kappa}_{\parallel}^{e} g_{\parallel}^{e(3)} g_{\parallel}^{e(3)} \\ & + \tilde{\sigma}_{\perp} \boldsymbol{g}_{\perp}^{(1)} \cdot \boldsymbol{g}_{\perp}^{(1)} + 2\tilde{\alpha}_{\perp} g_{\perp}^{(1)} \cdot \boldsymbol{g}_{\perp}^{e(3)} + \tilde{\kappa}_{\perp}^{e} \boldsymbol{g}_{\perp}^{e(3)} \cdot \boldsymbol{g}_{\perp}^{e(3)} \\ & + \tilde{\eta}_{\parallel}^{e} \boldsymbol{\nu}_{\parallel} : \boldsymbol{\nu}_{\parallel} + \tilde{\eta}_2^{e} \boldsymbol{\nu}_2 : \boldsymbol{\nu}_2 + \tilde{\eta}_4^{e} \boldsymbol{\nu}_4 : \boldsymbol{\nu}_4, \end{aligned}$$

$$\frac{\tau_i}{n_i} \sigma^i = \tilde{\kappa}_{\parallel}^{i} g_{\parallel}^{i(3)} g_{\parallel}^{i(3)} + \tilde{\kappa}_{\perp}^{i} \boldsymbol{g}_{\perp}^{i(3)} \cdot \boldsymbol{g}_{\perp}^{i(3)} + \tilde{\eta}_{\parallel}^{i} \boldsymbol{\nu}_{\parallel} : \boldsymbol{\nu}_{\parallel} + \tilde{\eta}_2^{i} \boldsymbol{\nu}_2 : \boldsymbol{\nu}_2 + \tilde{\eta}_4^{i} \boldsymbol{\nu}_4 : \boldsymbol{\nu}_4. \tag{4.6}$$

We now derive the conditions for the transport coefficients resulting from the *definite positive* character of the quadratic forms σ^e, σ^i. The present "transport form" is much simpler than the kinetic form (2.10). The ion entropy production is reduced by the symmetry requirement to a simple sum of squares. Therefore, the positivity conditions (Smirnov 1970) are simply

$$\tilde{\kappa}_{\parallel}^{i} \geqslant 0, \qquad \tilde{\kappa}_{\perp}^{i} \geqslant 0,$$

$$\tilde{\eta}_{\parallel}^{i} \geqslant 0, \qquad \tilde{\eta}_2^{i} \geqslant 0, \qquad \tilde{\eta}_4^{i} \geqslant 0. \tag{4.7}$$

The electronic entropy production is somewhat richer, because of the presence of the thermoelectric cross-coefficients. We therefore find the "diagonal" conditions

$$\tilde{\sigma}_{\parallel} \geqslant 0, \qquad \tilde{\sigma}_{\perp} \geqslant 0,$$

$$\tilde{\kappa}_{\parallel}^{e} \geqslant 0, \qquad \tilde{\kappa}_{\perp}^{e} \geqslant 0,$$

$$\tilde{\eta}_{\parallel}^{e} \geqslant 0, \qquad \tilde{\eta}_2^{e} \geqslant 0, \qquad \tilde{\eta}_4^{e} \geqslant 0, \tag{4.8}$$

but also the additional conditions

$$\tilde{\sigma}_{\parallel} \tilde{\kappa}_{\parallel}^{e} - \tilde{\alpha}_{\parallel}^2 \geqslant 0, \qquad \tilde{\sigma}_{\perp} \tilde{\kappa}_{\perp}^{e} - \tilde{\alpha}_{\perp}^2 \geqslant 0. \tag{4.9}$$

It can be checked directly that all these conditions are satisfied by the transport coefficients of table **5**.3.2 as a result of the corresponding conditions derived from the kinetic form of the entropy production. For instance,

$$\tilde{\sigma}_{\parallel}\tilde{\kappa}^{e}_{\parallel} - \tilde{\alpha}^{2}_{\parallel} = \begin{cases} \dfrac{1}{D^{e}_{13}}, & \text{13M approximation,} \\ \dfrac{c^{e}_{55}}{F^{e}_{135}}, & \text{21M approximation.} \end{cases} \tag{4.10}$$

The right-hand sides are positive, because of criteria (2.11)–(2.13).

Note that *the second law introduces no sign requirement on the individual thermoelectric coefficients* $\tilde{\alpha}_{\parallel}$, $\tilde{\alpha}_{\perp}$, $\tilde{\alpha}_{\wedge}$. Equation (4.9) can be satisfied indifferently by a positive or a negative $\tilde{\alpha}_B$: only its absolute value is constrained by the inequality. Actually, it was found (figs. **5**.4.1, **5**.4.2) that all these coefficients are negative for small values of $|x_\alpha|$, and that $\tilde{\alpha}_{\perp}$ and $\tilde{\alpha}_{\wedge}$ change sign for large $|x_\alpha|$. This behaviour is perfectly in agreement with thermodynamics.

Needless to say (but worth stressing!) that *thermodynamics requires nothing of the non-diagonal coefficients* $\tilde{L}_{\wedge}$, $\tilde{\eta}^{\alpha}_{1}$, $\tilde{\eta}^{\alpha}_{3}$. In particular, these may be positive or negative, without contradicting the second law. Actually, it was found (fig. **5**.4.2E) that

$$\tilde{\kappa}^{i}_{\wedge} \lessgtr 0,$$

and this may seem particularly shocking. It is, however, easily understood on intuitive grounds that this relation does not conflict with the second law. Consider first the parallel component of the ion heat flux,

$$\boldsymbol{g}^{i}_{\parallel} = -\kappa^{i}_{\parallel}(\boldsymbol{b}\cdot\nabla T_i)\boldsymbol{b}.$$

When the temperature gradient has a component along the magnetic field, the heat can only flow from hotter to colder regions, which implies $\kappa^{i}_{\parallel} \geqslant 0$. The same argument explains why $\kappa^{i}_{\perp} \geqslant 0$. But the $\wedge$-component of the heat flux is

$$\boldsymbol{g}^{i}_{\wedge} = -\kappa^{i}_{\wedge}(\boldsymbol{b}\wedge\nabla T_i).$$

This vector is perpendicular to the temperature gradient; it represents a *heat flux along a direction of constant temperature*. Such a purely reversible flow can go on in either direction, without contradicting the second law.

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7

Magnetohydrodynamics

7.1. *The classical hydrodynamical equations. Dissipative magnetohydrodynamics*

Our work of chapters 4–6 leads to a *complete set of hydrodynamical equations for a plasma in the presence of a homogeneous (or weakly inhomogeneous) magnetic field*. These *classical hydrodynamical equations* are valid whenever the collisions dominate the dynamics of the plasma and when the gradients are not too strong. These conditions were discussed in section **5**.1. An additional limitation must be added: the trajectories of the majority of the particles do not differ widely from their orbits in a straight, homogeneous magnetic field. The importance of this point will become clear and explicit in the forthcoming chapters.

The most general description in the present, classical framework is provided by the following equations. They were derived under the assumption that the charge density is everywhere negligible: $\sigma \approx 0$. We rewrite here these hydrodynamical equations for convenience, using dimensional quantities.

– *The continuity equation* (**4**.5.1):

$$\partial_t \rho = -\nabla \cdot (\rho \boldsymbol{u}). \tag{1.1}$$

– *The momentum balance equation* (**4**.5.2):

$$\partial_t \rho \boldsymbol{u} = -\nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} + P\boldsymbol{I}) + \frac{1}{c}\boldsymbol{j} \wedge \boldsymbol{B} + \nabla \cdot (\boldsymbol{\pi}^{\mathrm{e}} + \boldsymbol{\pi}^{\mathrm{i}}). \tag{1.2}$$

– *The electron temperature equation* (**4**.5.3):

$$\partial_t T_{\mathrm{e}} = -\hat{\boldsymbol{u}} \cdot \nabla T_{\mathrm{e}} - \tfrac{2}{3} T_{\mathrm{e}} \, \nabla \cdot \hat{\boldsymbol{u}} - \frac{2}{3n_{\mathrm{e}}} \boldsymbol{\pi}^{\mathrm{e}} : \nabla \boldsymbol{u}$$

$$- \frac{2}{3n_{\mathrm{e}}} \nabla \cdot \boldsymbol{q}^{\mathrm{e}} + \frac{1}{en_{\mathrm{e}}^2} \boldsymbol{j} \cdot \boldsymbol{R}^{\mathrm{ei}} - \frac{2}{\tau_{\mathrm{e}}} \mu (T_{\mathrm{e}} - T_{\mathrm{i}}), \tag{1.3}$$

where we used the notation

$$\hat{\boldsymbol{u}} \equiv \boldsymbol{u} - (en_e)^{-1}\boldsymbol{j}. \tag{1.4}$$

– *The ion temperature equation* (**4**.5.4):

$$\partial_t T_i = -\boldsymbol{u}\cdot\nabla T_i - \tfrac{2}{3}T_i\,\nabla\cdot\boldsymbol{u} - \frac{2}{3n_i}\boldsymbol{\pi}^i : \nabla\boldsymbol{u} - \frac{2}{3n_i}\nabla\cdot\boldsymbol{q}^i + \frac{2}{\tau_e}\mu Z(T_e - T_i). \tag{1.5}$$

– *The equation of state* (**3**.4.22):

$$P = n_e T_e + n_i T_i = \frac{\rho}{m_i}(ZT_e + T_i). \tag{1.6}$$

Equations (1.3)–(1.6) may also be combined into a single equation for the total pressure, which can be used instead of one (or both!) of the temperature equations, as will be seen below.

$$\partial_t P = -\boldsymbol{u}\cdot\nabla P - \tfrac{5}{3}P\nabla\cdot\boldsymbol{u} + \frac{5}{3e}\boldsymbol{j}\cdot\nabla T_e - \frac{2}{3en_e}\boldsymbol{j}\cdot\nabla(n_e T_e)$$

$$-\tfrac{2}{3}(\boldsymbol{\pi}^e + \boldsymbol{\pi}^i) : \nabla\boldsymbol{u} - \tfrac{2}{3}\nabla\cdot(\boldsymbol{q}^e + \boldsymbol{q}^i) + \frac{m_i}{Ze\rho}\boldsymbol{j}\cdot\boldsymbol{R}^{ei}. \tag{1.7}$$

These hydrodynamical equations are closed by using three additional sets of relations. The first group includes the electrodynamical moment equations:

– *The charge balance equation* (**4**.1.10):

$$\nabla\cdot\boldsymbol{j} = 0. \tag{1.8}$$

– *The linearized generalized Ohm law* [(**5**.1.7) with $\partial_t = 0$]:

$$e^2 n_e(\boldsymbol{E} + c^{-1}\boldsymbol{u}\wedge\boldsymbol{B}) + \nabla(n_e T_e) - \frac{e}{c}\boldsymbol{j}\wedge\boldsymbol{B} - e\boldsymbol{R}^{ei} = 0. \tag{1.9a}$$

We wrote the Ohm law in this form, in order to perform the forthcoming customary transformations. A completely equivalent, but fully explicit form is obtained by solving for the current, thus using (**5**.5.15),

$$\boldsymbol{j} = \sigma\cdot\hat{\boldsymbol{E}} - \boldsymbol{\alpha}\cdot\nabla T_e, \tag{1.9b}$$

where we recall that the effective electric field is defined as

$$\hat{\boldsymbol{E}} = \boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\wedge\boldsymbol{B} + \frac{1}{en_e}\nabla(n_e T_e). \tag{1.10}$$

The second group of closing equations defines the remaining dissipative quantities. It includes the transport equations (5.3.9), (5.3.16), (5.3.22),

$$\boldsymbol{q}^{\mathrm{e}} = T_{\mathrm{e}}\boldsymbol{\alpha}\cdot\hat{\boldsymbol{E}} - \boldsymbol{\kappa}^{\mathrm{e}}\cdot\nabla T_{\mathrm{e}}, \tag{1.11}$$

$$\boldsymbol{q}^{\mathrm{i}} = -\boldsymbol{\kappa}^{\mathrm{i}}\cdot\nabla T_{\mathrm{i}}, \tag{1.12}$$

$$\pi^{\alpha} = -\overleftrightarrow{\eta}^{\alpha} : \boldsymbol{\nu}. \tag{1.13}$$

We also need an expression for the *friction force* $\boldsymbol{R}^{\mathrm{ei}}$ (4.6.34),

$$\boldsymbol{R}^{\mathrm{ei}} = \frac{m_{\mathrm{e}}}{\tau_{\mathrm{e}}}\left(\frac{c_{11}^{\mathrm{e}}}{e}\boldsymbol{j} + \sqrt{\frac{2}{5}}\,\frac{c_{13}^{\mathrm{e}}}{T_{\mathrm{e}}}\boldsymbol{q}^{\mathrm{e}} + \cdots\right). \tag{1.14}$$

The third group of closing equations includes the *pre-Maxwell equations*

$$\partial_t \boldsymbol{B} = -c(\nabla \wedge \boldsymbol{E}), \tag{1.15}$$

$$\boldsymbol{j} = \frac{c}{4\pi}(\nabla \wedge \boldsymbol{B}), \tag{1.16}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{1.17}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sigma. \tag{1.18}$$

This set of equations is closed and may be taken as a basis for the *complete macroscopic description of the plasma*. It will be called here the set of *dissipative magnetohydrodynamical (MHD) equations*. It is, however, still too complicated for many practical applications. For this reason, it is customary to introduce additional simplifying assumptions in order to construct tractable *models* for describing the macroscopic plasma behaviour. It is the purpose of this chapter to review the most widely used of these models.

A first important remark concerns the general structure of the hydrodynamical equations of this set, i.e. eqs. (1.1)–(1.7). On the right-hand sides two groups of terms are clearly distinguished, corresponding to non-dissipative motions and to dissipative processes. We rewrite the equations in the following, half-schematic way, in order to exhibit this separation: the second class of terms is denoted by {DISS}:

$$\begin{aligned}
&\partial_t \rho = -\nabla\cdot(\rho\boldsymbol{u}),\\
&\partial_t \rho\boldsymbol{u} = -\nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u} + P\boldsymbol{I}) + c^{-1}\boldsymbol{j}\wedge\boldsymbol{B} + \{\mathrm{DISS}\},\\
&\partial_t T_{\alpha} = -\boldsymbol{u}\cdot\nabla T_{\alpha} - \tfrac{2}{3}T_{\alpha}\,\nabla\cdot\boldsymbol{u} + \{\mathrm{DISS}\},\\
&\partial_t P = -\boldsymbol{u}\cdot\nabla P - \tfrac{5}{3}P\,\nabla\cdot\boldsymbol{u} + \{\mathrm{DISS}\}.
\end{aligned} \tag{1.19}$$

The main difference between the two classes of terms is their order of magnitude with respect to the parameter λ_H introduced in (**5**.1.3),

$$\begin{cases} \text{Non-dissipative terms: Order } \lambda_H, \\ \text{Dissipative terms: Order } \lambda_H^2. \end{cases} \tag{1.20}$$

In the collision-dominated classical regime, one is tempted to consider that the second group of terms is small compared to the first, and may therefore be neglected. However, this procedure is sometimes dangerous, because the mathematical nature of the dissipative terms is quite different from the first. As a result, even a very small amount of dissipation may lead to phenomena dramatically different from the predictions of the non-dissipative theory. For instance, it is clear that if the dissipation is to be regarded as a small correction, it must be treated by the methods of "singular perturbation theory", because the corresponding terms are of higher differential order than the non-dissipative terms. As a result, this will lead to the formation of sheaths, where the gradients may become very steep. It is therefore indispensable to put a big warning sign in front of the forthcoming discussion:

A given set of approximations is never universally valid, even for a given range of physical parameters. Their relevance must be checked in detail for every particular problem.

As a first, rather trivial example, we consider the following assumption, which is very common in the literature (see e.g. Goedbloed 1983):

$$|\boldsymbol{u}^e - \boldsymbol{u}^i| \ll |\boldsymbol{u}|,$$

or

$$|\boldsymbol{j}| \ll en_e|\boldsymbol{u}|. \tag{1.21}$$

The assumption appears reasonable, in general, because $\boldsymbol{u}$ is a hydrodynamical quantity (thus of order λ_H^0), whereas $\boldsymbol{j}$ is non-hydrodynamical, hence of order λ_H. If this approximation is accepted, then the quantity $\hat{\boldsymbol{u}}$ in the electron temperature equation (1.3), (1.4) is simply replaced by $\boldsymbol{u}$, and the term $\boldsymbol{j} \cdot \nabla T_e$ may be dropped in eq. (1.7). On the same basis, the term $\boldsymbol{j} \wedge \boldsymbol{B}$ could be dropped in the Ohm law (1.9a)! The consequences of these features will be described in the next section.

Here we only mention what happens when the hydrodynamical equations are applied to the study of a (static) equilibrium configuration. In this case, one considers (by definition!) situations in which $\boldsymbol{u} = 0$, but $\boldsymbol{j} \neq 0$. Hence approximation (1.21) becomes completely wrong!

7.2. Resistive magnetohydrodynamics

The first simplification of the dissipative MHD equations, which is very frequently introduced in the plasma physics literature, will be presently discussed (see, e.g. Braginskii 1965, Golant et al. 1980, Freidberg 1982, Goedbloed 1983). It consists of reducing the dissipative processes to their simplest expression. The approximation consists of *neglecting the heat fluxes* q^α *as well as the momentum fluxes, i.e. the dissipative pressure tensors* π^α in all the equations. The only dissipation mechanism left in this picture is the electrical resistivity. Therefore, the resulting model is called *resistive magnetohydrodynamics* (*MHD*).

The procedure can be reformulated as follows in the general framework of chapter 4. Referring to eq. (**4**.3.8), the deviations χ^α from the local plasma equilibrium are simply put equal to zero. Equivalently, (see **4**.3.11), *all the Hermitian moments* $h^{\alpha(p)}_{\ldots}$ *are declared negligible*. This could be called the *5-moment (5M) approximation* and is, indeed, the crudest approximation which still retains some form of dissipation. The origin of the latter is the difference between the local velocities of the species, $\boldsymbol{u}^e - \boldsymbol{u}^i \neq 0$, which implies the existence of an electric current.

We now discuss this assumption. Its first consequence is the replacement of eqs. (1.11)–(1.13) by

$$\boldsymbol{q}^e \equiv \boldsymbol{q}^i \equiv 0, \qquad \pi^e \equiv \pi^i \equiv 0. \tag{2.1}$$

This, in turn, implies

$$\boldsymbol{\alpha} \equiv 0, \qquad \kappa^\alpha \equiv 0, \qquad \overleftrightarrow{\eta}^\alpha \equiv 0. \tag{2.2}$$

Let us consider in some detail one of these relations, say $\kappa^\alpha = 0$; the discussion of the others is quite similar. The vanishing of the tensor κ^α implies three relations:

$$\kappa^\alpha_{\parallel} \equiv 0, \qquad \kappa^\alpha_{\perp} = 0, \qquad \kappa^\alpha_{\wedge} = 0. \tag{2.3}$$

We know from section **5**.6 that, *if the magnetic field is sufficiently strong*, the two conditions $\kappa^\alpha_{\perp} \equiv 0$, $\kappa^\alpha_{\wedge} \equiv 0$ can be easily satisfied; however, the smallness of $\kappa^\alpha_{\parallel}$ poses an altogether different problem. Indeed, looking at table **5**.3.3, we see that there are two ways of satisfying this condition. One is to assume that *the density* n_α *and/or the temperature* T_α, *or simply the pressure* P_α, *are very small*. This is indeed the case in many problems of *astrophysical interest* (e.g. the solar wind, the planetary magnetospheres,...), but not in the fusion context. Another possibility is to assume that *the collisional relaxation time is*

very short, $\tau_\alpha \approx 0$ (or that the collision frequency $\nu_\alpha = \tau_\alpha^{-1}$ is very high). This implies a plasma regime which is *very strongly dominated by collisions*. This condition is, again, not very likely to be satisfied in a fusion context.

The next simplification appears in the expression of the friction force (1.14). Upon setting all the Hermitian moments equal to zero, this force become simply proportional to the electric current (remember that $c_{11}^{e} = 1$, see table 5.3.1):

$$\boldsymbol{R}^{ei} \cong \frac{m_e}{e\tau_e}\boldsymbol{j}. \tag{2.4}$$

Upon substituting this expression into the Ohm law (1.9a), we get

$$\frac{m_e}{\tau_e}\boldsymbol{j} + \frac{e}{c}\boldsymbol{j} \wedge \boldsymbol{B} = e^2 n_e(\boldsymbol{E} + c^{-1}\boldsymbol{u} \wedge \boldsymbol{B}) + e\,\nabla(n_e T_e). \tag{2.5}$$

Its solution, obtained according to the procedure of section 5.3 is

$$\boldsymbol{j} = \boldsymbol{\sigma} \cdot \left[\boldsymbol{E} + c^{-1}\boldsymbol{u} \wedge \boldsymbol{B} + (en_e)^{-1}\nabla(n_e T_e)\right]. \tag{2.6}$$

The conductivity tensor $\boldsymbol{\sigma}$ has the usual form (5.5.22) and its individual components are given by (5.3.11),

$$\sigma_A = \frac{e^2 n_e}{m_e}\tau_e\,\tilde{\sigma}_A, \quad A = \parallel, \perp, \wedge, \tag{2.7}$$

with

$$\tilde{\sigma}_\parallel = 1, \qquad \tilde{\sigma}_\perp = \frac{1}{1 + x_e^2}, \qquad \tilde{\sigma}_\wedge = -\frac{x_e}{1 + x_e^2}, \tag{2.8}$$

where $x_e = \Omega_e \tau_e$.

Equation (2.6) is usually employed in MHD for expressing the electric field in terms of the current [which is determined by the Maxwell equation (1.16)]. Equation (2.6) is therefore inverted, with the result

$$\boldsymbol{E} + \frac{1}{c}\boldsymbol{u} \wedge \boldsymbol{B} + \frac{1}{en_e}\nabla(n_e T_e) = \boldsymbol{\sigma}^{-1} \cdot \boldsymbol{j}, \tag{2.9}$$

where $\boldsymbol{\sigma}^{-1}$ is the *resistivity tensor*. [This (inverse) transport tensor is commonly denoted by $\boldsymbol{\eta}$ in the literature, but this notation leads here to confusion with the viscosity tensor].

In the MHD literature, even these equations are considered too complex; two additional simplifications are introduced. First, it is assumed that *the ion Larmor radius r_{Li} is much smaller than the hydrodynamic length*,

$$\frac{r_{Li}}{L_H} \ll 1. \tag{2.10}$$

This is a very reasonable assumption, which is easily satisfied both in fusion regimes and in astrophysics: it is nothing other than the expression of the *drift approximation* (see **1**.5.26). Under this assumption, the pressure gradient term can be compared to the $\boldsymbol{u} \wedge \boldsymbol{B}$ term on the left-hand side of (2.9) (Freidberg 1982),

$$\frac{(en_e)^{-1}|\nabla(n_eT_e)|}{c^{-1}|\boldsymbol{u}\wedge\boldsymbol{B}|} \approx \frac{(m_em_iL_H)^{-1}Ze\rho T_e}{(m_em_ic)^{-1}Ze^2\rho uB} = \frac{1}{L_H}\frac{m_ic}{eB}\frac{T_e}{m_i}\frac{1}{u}$$

$$= \frac{r_{Li}}{L_H}\left(\frac{T_e}{T_i}\right)^{1/2}\frac{c_s}{u},$$

where $c_s = (T_e/m_i)^{1/2}$ is the well-known ion-acoustic velocity. If, as usual, $u \approx c_s$, and if T_e is not very different from T_i, the ratio of the two terms is of order (r_{Li}/L_H). Thus, *the electron pressure gradient term can be neglected in eq.* (2.8).

Next, it is usually assumed that condition (1.21) can be used in the Ohm law (2.5) in order to *suppress the term $\boldsymbol{j} \wedge \boldsymbol{B}$*. The equation then reduces to

$$\boldsymbol{j} = \frac{e^2n_e}{m_e}\tau_e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\wedge\boldsymbol{B}\right),$$

or

$$\boldsymbol{j} = \sigma_{\parallel}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\wedge\boldsymbol{B}\right). \tag{2.11}$$

Its inversion is trivial,

$$\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}\wedge\boldsymbol{B} = \frac{1}{\sigma_{\parallel}}\boldsymbol{j}. \tag{2.12}$$

Equations (2.11), (2.12) are found in all classical texts on MHD (Alfvén 1950, Cowling 1975, Freidberg 1982, Kulsrud 1983, etc.). This approximation scheme calls for two remarks.

(a) The clearest misdeed of the brutal application of approximation (1.21) is the *loss of the anisotropy* of the conductivity tensor. This has serious consequences: by using (2.11) instead of (2.6) for plasmas in the presence of strong magnetic fields (remember 2.10), *the perpendicular components of the electric current* (for a given electric field) *are grossly overestimated*. Indeed, we know from sections **5**.4 and **5**.6 that $\sigma_\perp$ and $\sigma_\wedge$ decay rapidly to zero for strong magnetic fields.

If, on the other hand, the Ohm law is used in the form (2.9), the use of (2.12) for determining the electric field also leads to inconsistencies. The parallel component is given correctly; as for the perpendicular components, we find from (2.9), combined with (**5**.5.28),

$$e_1 \cdot \boldsymbol{E}' = (\sigma^{-1})_\perp e_1 \cdot \boldsymbol{j} + (\sigma^{-1})_\wedge e_2 \cdot \boldsymbol{j}, \tag{2.13}$$

where $\boldsymbol{E}' \equiv \boldsymbol{E} + \mathrm{c}^{-1}\boldsymbol{u} \wedge \boldsymbol{B}$, and e_1, e_2 are basis vectors perpendicular to $\boldsymbol{B}$. We know from the discussion in section **5**.7(c) that, in the limit of a strong magnetic field,

$$(\sigma^{-1})_\perp \sim \frac{m_\mathrm{e}}{e^2 n_\mathrm{e} \tau_\mathrm{e}}.$$

Hence, the first, diagonal term in (2.13) would be precisely consistent with $\sigma_\parallel^{-1}$ (2.8). * However, the second, non-diagonal term becomes very large, growing linearly with $\Omega_\mathrm{e}\tau_\mathrm{e}$!

We thus see that the use of the "innocent" approximation (1.21) completely changes the character of the solution. Thus, eq. (2.11) has, at best, the status of a *model equation*, possessing the advantage of mathematical simplicity. It has not been derived as a well-controlled approximation to the exact equations. The validity of its consequences should therefore be checked in each particular case.

(b) Suppose we accept the isotropic model for the conductivity: we then derive eq. (2.11) *with value* (2.8) *for the parallel conductivity*, i.e. $\tilde{\sigma}_\parallel = 1$. But we know from table **5**.4.1 that this value is *too small by a factor* ≈ 2 as compared to the exact value $\tilde{\sigma}_\parallel = 1.953$. One could, of course, simply "patch up" the model by inserting the exact value of $\sigma_\parallel$ into (2.11). This, however, is an inconsistent procedure, because the 13M value (and, a fortiori, the higher approximations) of the conductivity results from a *coupling* between the electric current and the heat flux leading to the transport equations (1.9b), (1.11). But in the present approximation the heat flux was put equal to zero!

* Actually, even this "consistency" is fortuitous. We know from (**5**.7.24) that the exact perpendicular resistivity is about twice the parallel one! The reason of the equality here is the incorrect value of $\sigma_\parallel$, see (b) below.

We now turn to the pressure equation (1.7). With the approximations (2.1), (2.4), the last three terms of the right-hand side reduce to $\boldsymbol{j}\cdot\boldsymbol{R}^{\mathrm{ei}} \sim \sigma_{\parallel}^{-1}\, j^2$. Consider now the term $\boldsymbol{j}\cdot\nabla T_{\mathrm{e}}$: this term is assumed to be small compared to the non-dissipative term $\boldsymbol{u}\cdot\nabla T_{\mathrm{e}}$, on the basis of (1.21). However, in the general case, its value is comparable to a dissipative term contained in $\boldsymbol{j}\cdot\boldsymbol{R}^{\mathrm{ei}}$: $\boldsymbol{j}\cdot\boldsymbol{q}^{\mathrm{e}} \sim -\boldsymbol{j}\cdot\boldsymbol{\kappa}^{\mathrm{e}}\cdot\nabla T_{\mathrm{e}}$. Thus, if the complete dissipative contribution is retained, the term $\boldsymbol{j}\cdot\nabla T_{\mathrm{e}}$ cannot be neglected; if, however, the 5M approximation is accepted, its neglect is justified. A similar argument applies to the term $\boldsymbol{j}\cdot\nabla(n_{\mathrm{e}}T_{\mathrm{e}})$ in (1.7).

The neglect of the terms $\boldsymbol{j}\cdot\nabla T_{\mathrm{e}}$ and $\boldsymbol{j}\cdot\nabla(n_{\mathrm{e}}T_{\mathrm{e}})$ introduces a new feature into the model: *none of the temperatures T_{e} or T_{i} appears any longer separately in the equations of the model*: they only contribute to the evolution in the combination (1.6). We thus no longer need *two* equations (1.3), (1.5) for a closed description: *the unique pressure balance equation is sufficient for the closure* (in spite of the existence of two different temperatures, $T_{\mathrm{e}} \neq T_{\mathrm{i}}$). This equation reduces to

$$\partial_t P + \boldsymbol{u}\cdot\nabla P = -\tfrac{5}{3}P\,\nabla\cdot\boldsymbol{u} + \frac{1}{\sigma_{\parallel}}j^2. \tag{2.14}$$

The last term is this equation clearly describes the dissipation of energy through the *Joule effect*.

The simplified equations which come out of the previous discussion are now combined into an equivalent set, from which some of the quantities are explicitly eliminated. The strategy goes as follows.

– The *electric current* $\boldsymbol{j}$ is obtained from (1.16) [which automatically satisfies (1.8)].

– The *electric field* $\boldsymbol{E}$ is determined from (2.12).

– As a result, the momentum balance equation (1.2), combined with (1.1), yields the *equation of motion*,

$$\rho(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P + \frac{1}{c}\boldsymbol{j}\wedge\boldsymbol{B} = -\nabla P - \frac{1}{4\pi}\boldsymbol{B}\wedge(\nabla\wedge\boldsymbol{B}),$$

which, in turn, is combined with (1.16).

– Finally, the magnetic field obeys eq. (1.15), which is combined with (2.12) and (1.17),

$$\partial_t\boldsymbol{B} = -c\nabla\wedge\boldsymbol{E} = \nabla\wedge(\boldsymbol{u}\wedge\boldsymbol{B}) + \frac{c^2}{4\pi\sigma_{\parallel}}\nabla^2\boldsymbol{B}.$$

The outcome of this discussion is a set of *four closed equations for the three hydrodynamical quantities* ρ, $\boldsymbol{u}$, P *and for the magnetic field* $\boldsymbol{B}$. These equations

Table 2.1.
The equations of resistive MHD.

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\boldsymbol{u} \tag{R.1}$$

$$\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nabla P - \frac{1}{4\pi}\boldsymbol{B}\wedge(\nabla\wedge\boldsymbol{B}) \tag{R.2}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\tfrac{5}{3}P\nabla\cdot\boldsymbol{u} + \frac{c^2}{16\pi^2\sigma_{\parallel}}|\nabla\wedge\boldsymbol{B}|^2 \tag{R.3}$$

$$\partial_t\boldsymbol{B} = \nabla\wedge(\boldsymbol{u}\wedge\boldsymbol{B}) + \frac{c^2}{4\pi\sigma_{\parallel}}\nabla^2\boldsymbol{B} \tag{R.4}$$

Constraint

$$\nabla\cdot\boldsymbol{B} = 0 \tag{R.5}$$

Definitions

$$\boldsymbol{E} = -\frac{1}{c}(\boldsymbol{u}\wedge\boldsymbol{B}) + \frac{c}{4\pi\sigma_{\parallel}}(\nabla\wedge\boldsymbol{B}) \tag{R.6}$$

$$\boldsymbol{j} = \frac{c}{4\pi}(\nabla\wedge\boldsymbol{B}) \tag{R.7}$$

$$\sigma = -\frac{1}{4\pi c}\nabla\cdot(\boldsymbol{u}\wedge\boldsymbol{B}) \tag{R.8}$$

Note

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \boldsymbol{u}\cdot\nabla \tag{R.9}$$

are augmented by a constraint for the magnetic field and by three "auxiliary" relations *defining* the electric field $\boldsymbol{E}$, the electric current $\boldsymbol{j}$ and the electric charge density σ. These equations are brought together in table 2.1, where they are renumbered for easy reference.

The outstanding feature here is that the parallel electrical conductivity (or its inverse, the resistivity) is the only remaining transport coefficient describing the dissipation in this model. This justifies its name: the *resistive magnetohydrodynamics* (or simply *resistive MHD*). It is a widely used model in the plasma physics literature.

7.3. *Ideal magnetohydrodynamics*

Even the equations of table 2.1 are often considered too complicated for a detailed study. An additional simplification is postulated in order to suppress completely the remaining dissipative terms. *Formally*, this postulate is formulated by saying that the plasma has an *almost infinite parallel conductivity* ($\sigma_{\parallel} \approx \infty$) (or an *almost vanishing parallel resistivity*). The term involving $\sigma_{\parallel}$ in eqs. (R.3), (R.4) and (R.6) of table 2.1 can then be neglected and the equations of table 3.1 are obtained (see below). They are properly called the equations of *ideal magnetohydrodynamics* (or *ideal MHD*). They constitute the most popular model for the study of the macroscopic plasma dynamics, both in astrophysics and in fusion theory.

The assumption justifying their use must, however, be discussed more carefully. The mere statement that "$\sigma_{\parallel}$ is large" has no intrinsic meaning, because $\sigma_{\parallel}$ is not a dimensionless quantity.

An even less favourable point appears when we recall that the other transport coefficients $\kappa_{\parallel}^{\alpha}$, $\alpha_{\parallel}$, $\eta_{\parallel}^{\alpha}$, were discarded on the basis of the assumption that the plasma dynamics is strongly dominated by collisions, i.e. $\tau_{\alpha} \approx 0$. If this argument is consistently pursued, it leads to the conclusion that $\sigma_{\parallel}$ (which is proportional to τ_e, see table **5**.3.3) would be very *small*, rather than very large, as assumed above!

The correct argument for the justification of ideal MHD comes from a comparison of the two terms on the right-hand side of each of the equations (R.3), (R.4) and (R.6) of table 2.1. Consider, for instance, eq. (R.4). We obtain the following estimate in terms of the hydrodynamic length scale L_H, introduced in section 7.2,

$$\frac{|\nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B})|}{|(c^2/4\pi\sigma_{\parallel}) \nabla^2 \boldsymbol{B}|} \approx \frac{uBL_H^{-1}}{(c^2/4\pi\sigma_{\parallel}) BL_H^{-2}} \approx \frac{4\pi}{c^2} \sigma_{\parallel} uL_H.$$

The right-hand side is a dimensionless number which will be called the *magnetic Reynolds number* R_m,

$$R_m = \frac{4\pi}{c^2} \sigma_{\parallel} uL_H. \tag{3.1}$$

This name is not arbitrary. The ordinary, hydrodynamic Reynolds number

$$Re = \frac{\rho}{\eta_{\parallel}} uL_H \tag{3.2}$$

is a measure of the relative importance of the inertial, *convective* terms, compared to the *dissipative* viscous terms in the Navier–Stokes equation (1.2).

Similarly, R_m measures the relative size of the *convective* terms, relative to the *dissipative* terms in the Maxwell equation (R.4). The role of the kinematic viscosity ($\eta_{\parallel}/\rho$) is played by the resistivity, or rather the combination ($c^2/4\pi\sigma_{\parallel}$) in the magnetic Reynolds number.

Our conclusion concerning eq. (R.4) is

$$\frac{|\nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B})|}{|(c^2/4\pi\sigma_{\parallel})\nabla^2 \boldsymbol{B}|} \approx R_m. \tag{3.3}$$

A similar estimate in (R.6) shows that

$$\frac{|c^{-1}(\boldsymbol{u} \wedge \boldsymbol{B})|}{|(c/4\pi\sigma_{\parallel})(\nabla \wedge \boldsymbol{B})|} \approx R_m, \tag{3.4}$$

Finally, in eq. (R.3), we find [by estimating the pressure term from (R.2) with $\boldsymbol{u} = 0$]

$$\frac{|\frac{5}{3}P \nabla \cdot \boldsymbol{u}|}{|(c^2/16\pi^2\sigma_{\parallel})|\nabla \wedge \boldsymbol{B}|^2|} \approx \tfrac{5}{3}R_m. \tag{3.5}$$

We thus conclude that *the condition of validity of ideal MHD*, formulated intrinsically in dimensionless form is

$$R_m \gg 1. \tag{3.6}$$

From (3.1) we see that the magnetic Reynolds number can be made large, even for a small parallel conductivity, by considering situations where the velocity and/or the length scale is vary large. In practice, (3.6) is best satisfied for real plasmas when L_H is very large. Priest (1982) very adequately says "[*ideal MHD*] *is sometimes referred to as the infinite-conductivity limit, but it would be better called the large length-scale limit...*". As a consequence, ideal MHD is particularly well suited in *astrophysical problems*, because of the size of typical astronomical distances.

In laboratory plasmas, R_m is most often smaller than one. A rather careful analysis performed by Freidberg (1982) leads to the disappointing conclusion that the domain of parameters for which ideal MHD is valid falls entirely outside of the thermonuclear fusion domain *.

* We refer to the original paper by Freidberg (1982) for the consideration of an alternative model ("Perpendicular MHD") whose validity domain is closer to fusion conditions.

In order to bring out more clearly the role of the electrical conductivity, we briefly compare the two extreme cases.

(a) *The strongly resistive case*: $R_m \ll 1$.

In this case, eq. (R.4) of table 2.1 reduces to

$$\partial_t \boldsymbol{B} = D_m \nabla^2 \boldsymbol{B}, \tag{3.7}$$

where we introduced the coefficient

$$D_m = \frac{c^2}{4\pi\sigma_{\parallel}}. \tag{3.8}$$

Equation (3.7) is a typical *diffusion equation*, and D_m is called the *magnetic diffusion coefficient* (or *magnetic diffusivity*). As a result, a magnetic field configuration initially peaked, say, around the origin, diffuses away and brings about a decay of the magnetic field intensity. The characteristic time associated with this process is clearly:

$$\tau_D = \frac{L_H^2}{D_m} = \frac{4\pi}{c^2} L_H^2 \sigma_{\parallel}. \tag{3.9}$$

It turns out that in a tokamak, this time scale is much longer than the other relevant time scales connected, say, with the growth rates of MHD instabilities. For this reason, the resistive effects have been considered unimportant in the early history of fusion physics. This point of view was radically changed when Furth et al. (1963) showed that a finite resistivity may give rise to instabilities growing on a much faster time scale (see also Hazeltine and Meiss 1985), such as the celebrated *tearing modes*. This subject will not be treated in the present volume. We only mention it here in order to show how careful one must be before drawing conclusions from the various models.

The diffusive decay of the magnetic field intensity implies that the magnetic energy is transformed into thermal energy. This is clearly apparent in the energy equation (R.3), which reduces here to

$$\frac{dP}{dt} = \frac{1}{4\pi} D_m |\nabla \wedge \boldsymbol{B}|^2. \tag{3.10}$$

The right-hand side is definite positive quantity. This equation therefore describes the monotonous increase of pressure, hence of thermal energy, due to Joule heating.

(b) *The ideal MHD limit*: $R_m \gg 1$.

This case corresponds to the equations of table 3.1. All the dissipative effects have disappeared in this model. Moreover, the ideal MHD is a typical

Table 3.1
The equations of ideal MHD.

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\nabla\cdot(\rho \boldsymbol{u}) \tag{I.1}$$

$$\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nabla P - \frac{1}{4\pi}\boldsymbol{B}\wedge(\nabla\wedge\boldsymbol{B}) \tag{I.2}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\tfrac{5}{3}P\nabla\cdot\boldsymbol{u} \tag{I.3}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\wedge(\boldsymbol{u}\wedge\boldsymbol{B}) \tag{I.4}$$

Constraint

$$\nabla\cdot\boldsymbol{B} = 0 \tag{I.5}$$

Definitions

$$\boldsymbol{E} = -\frac{1}{c}(\boldsymbol{u}\wedge\boldsymbol{B}) \tag{I.6}$$

$$\boldsymbol{j} = \frac{c}{4\pi}(\nabla\wedge\boldsymbol{B}) \tag{I.7}$$

$$\sigma = -\frac{1}{4\pi c}\nabla\cdot(\boldsymbol{u}\wedge\boldsymbol{B}) \tag{I.8}$$

one-fluid model: only the *global* hydrodynamical quantities ρ, $\boldsymbol{u}$, P enter eqs. (1.1)–(1.4). This is a rather curious feature, because we have *not* supposed that the temperatures of the two components are equal! Its origin is in the neglect of the dissipative terms, whose description requires the more refined two-fluid picture (see the discussion leading to eq. 2.14).

This feature clearly appears in the relation between the energy equation (1.3) and the entropy balance. We stressed in chapter 6 the difficulty of defining a *total* entropy in the case of a two-temperature plasma. However, after all the idealizations which have led to the present model, it is no longer possible to distinguish the two components in the equations of table 3.1. Therefore, the only definition of the entropy consistent with this model is a global one, proceeding as in the traditional method (de Groot and Mazur 1984) from the Gibbs equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}s = \frac{\mathrm{d}}{\mathrm{d}t}e + P\frac{\mathrm{d}}{\mathrm{d}t}\rho^{-1}. \tag{3.11}$$

If the internal energy per particle, e, is expressed in terms of the pressure and the density, we have

$$e = \frac{3}{2}\frac{P}{\rho} = \frac{P}{(\gamma - 1)\rho}. \tag{3.12}$$

We have included the second expression, though it is not really necessary; it involves the well-known ratio γ of the specific heats at constant pressure and constant volume. For our fully ionized plasma, $\gamma = \frac{5}{3}$. Combining eqs. (3.11) (3.12) with eqs. (I.1), (I.3) of table 3.1, we easily find

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0, \tag{3.13}$$

which shows, not surprisingly, that the flows described by the ideal MHD are *isentropic*, or *adiabatic*. In other words, the entropy in every element of fluid following the motion remains constant in time. There is no entropy production, because all the dissipative processes have been neglected in this model. Another form of (I.3), which is very often found in the literature, is obtained by a combination with (I.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{P}{\rho^{5/3}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{P}{\rho^{\gamma}} = 0. \tag{3.14}$$

This equation expresses the well-known fact that in an adiabatic flow the ratio P/ρ^{γ} remains constant in every moving volume element.

We now turn to the consideration of eq. (I.4), which determines the evolution of the magnetic field. Its physical content can be stated as follows: *the motion of the magnetic field is solely due to its convection by the fluid motion*. This is the main characteristic of the ideal MHD model. The statement is made more precise by the following *frozen-flux theorem* due to Alfvén (1950). *In the ideal MHD model, the magnetic field lines behave as if they move with the plasma*. We sketch its proof, following closely Priest (1982).

Consider a closed line C moving with the plasma (i.e., each of its points moves with the local velocity of the fluid at that point). This line is the boundary of a region S (see fig. 3.1A). During a time δt an element $\delta \boldsymbol{s}$ of the contour sweeps out an area $\boldsymbol{u}\,\delta t \wedge \delta \boldsymbol{s}$. The magnetic flux passing through this area is

$$\boldsymbol{B}\cdot(\boldsymbol{u}\delta t \wedge \delta \boldsymbol{s}) = -\delta t(\boldsymbol{u} \wedge \boldsymbol{B})\cdot \delta \boldsymbol{s}.$$

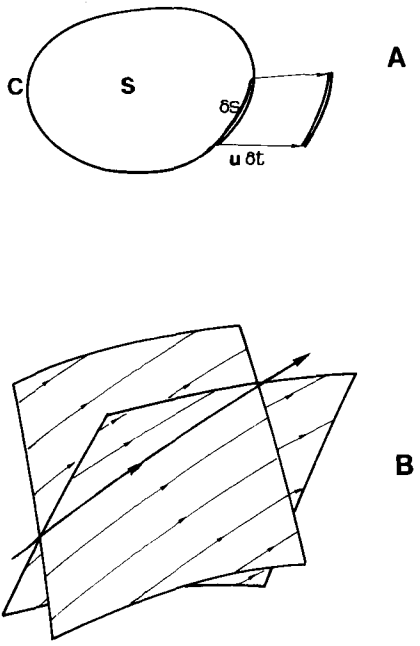

Fig. 3.1. Proof of the frozen field theorem.

Consider now the total flux through the surface S:

$$\Phi(t) = \int_{S(t)} \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B}(t).$$

Its rate of change in time is due to two causes: the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ changes in time at each point, and the boundary $S(t)$ moves with the fluid. Hence

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \int_S \mathrm{d}\boldsymbol{S} \cdot \partial_t \boldsymbol{B} - \int_C \mathrm{d}\boldsymbol{s} \cdot (\boldsymbol{u} \wedge \boldsymbol{B}).$$

The second integral is transformed into a surface integral by using Stokes' theorem,

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \int_S \mathrm{d}\boldsymbol{S} \cdot \left(\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) \right) = 0.$$

The right-hand side vanishes identically, on account of (I.4). Thus, *the flux across a surface moving with the plasma remains constant in time.*

Consider now two intersecting *magnetic surfaces*, i.e. surfaces to which the magnetic field lines are everywhere tangent (fig. 3.1B). Their intersection is thus a field line. By definition, the flux through each of these surfaces is zero. If the magnetic surfaces move with the plasma, the flux through each of them must remain zero, by the previous theorem. Thus each one remains a magnetic surface, and their intersection remains a magnetic field line. Hence, *the magnetic field lines move with the plasma.*

In Priest's words: "*One refers to field lines being frozen into the plasma; plasma can move along field lines, but, in motion perpendicular to them, either the field lines are dragged with the plasma, or the field lines push the plasma*". This peculiar behaviour is the most dramatic consequence of the assumption $R_m \gg 1$. If the magnetic Reynolds number has a smaller value, the magnetic field lines may slip through the plasma or (conversely but more significantly): the plasma may slip through the magnetic field.

We now turn to the last equation affected by the ideal MHD approximation: eq. (I.6) of table 3.1, which is the "remainder" of the Ohm law. Here we clearly see the change of status of the various equations. The Ohm law no longer provides a relation between current and electric field, because the former dropped out in the limit $R_m \gg 1$. Thus the Ohm law completely determines the electric field $\boldsymbol{E}$, which is produced by the motion of the plasma through the magnetic field. The electric current is determined separately by the Ampère law (I.7). As for the Poisson equation, it is no longer needed for the determination of the electric field. It becomes a completely marginal equation (I.8) that determines the electrical charge density which, in turn, is not needed anywhere in the equations.

An important consequence of (I.6) is the fact that the electric field is everywhere perpendicular to both the magnetic field and the plasma velocity:

$$\boldsymbol{E} \cdot \boldsymbol{B} = 0, \qquad \boldsymbol{E} \cdot \boldsymbol{u} = 0. \tag{3.15}$$

In particular, if in a given situation, an electric field component parallel to $\boldsymbol{B}$ is found, this is a signature of a finite resistivity effect.

7.4. *Magnetohydrodynamics, astrophysics and fusion. The strategy of fusion theory*

We have presented in the previous sections the succession of arguments leading from the "exact" macroscopic description of the plasma, i.e. the

dissipative MHD equations, to the *ideal MHD model* of section 7.3. We endeavoured to be as critical as possible, trying to pin down every assumption made in this process. The result is not very satisfactory: The approximations are not all really justified and are even sometimes inconsistent.

The motivation of this process is the same as the consideration of an *ideal fluid*, modelled by the *Euler equations* in ordinary hydrodynamics, as an approximation of the real, dissipative fluid described by the *Navier–Stokes equations*. It appears, though, that the coupling with the electromagnetic field makes this conceptual modelization more difficult in the case of a plasma. Kulsrud (1983) very properly defines the status of ideal MHD: "[*The ideal MHD equations*] *are clearly an approximation to the true plasma equations, but they have so many nice properties that they are the preferred set for describing macroscopic plasma phenomena*".

Not surprisingly, ideal MHD was invented by the astrophysicist H. Alfvén, and the most satisfactory expositions of this discipline are due to the astrophysicists Cowling (1957), Parker (1979) and Priest (1982). It is indeed in the field of astrophysics that most situations lying really within the validity domain of ideal MHD are found. The reason is the enormous size of the characteristic length scales in these cases. Let us quote, among the problems dealt with in astrophysical MHD: the structure of cosmical (interplanetary, interstellar, intergalactic...) magnetic fields, the solar wind, the solar flares and other aspects of the solar activity, the structure of planetary magnetospheres, etc. In these problems, the collisional dissipation is absent (almost by definition), therefore they fall outside of the scope of this book. In some cases, however, when a strong turbulence develops for various reasons, an "anomalous transport" may appear and produce important effects. Such problems will however not be treated in the present volume.

With the emergence of the idea of controlled thermonuclear fusion as a terrestrial source of energy, the physicists began to acquaint themselves with the findings of the astrophysicists (it is not a fortuitous coincidence that one of the earliest promotors of controlled fusion was the *astrophysicist* L. Spitzer Jr.). The most attractive idea that came from ideal MHD was Alfvén's frozen field theorem. Indeed, it suggested that if a "closed" magnetic field configuration could be realized, this configuration might act as a "trap" or "bottle" for the plasma. The latter could thus be confined in a finite region and be heated to extremely high temperatures; the fusion reactions could then proceed "at ease" in this environment and produce useful energy.

It was realized afterwards that the picture was too strongly idealized. As shown in the previous section, the condition $R_m \gg 1$ is very hard to satisfy in terrestrial laboratory conditions because of the relative smallness of the length scales involved. The plasma would necessarily "leak" through the magnetic field. The next, less ambitious goal became the control and optimization of

these leaks, in order to confine the plasma for a finite time, long enough to allow the production of a sufficient amount of fusion energy before breakdown. In this perspective, *transport theory* becomes a key discipline for understanding and controlling the various leaking mechanisms that may exist in a plasma.

Among the many existing expositions of MHD theory from the fusion viewpoint, we quote the following texts: Braginskii (1965), Shafranov (1966), Kadomtsev (1966), Soloviev and Shafranov (1970), Bateman (1978), Golant et al. (1980), Freidberg (1982), Goedbloed (1983), Kulsrud (1983).

The strategy set up in fusion theory proceeds (schematically) in several steps.

(a) *Equilibrium theory*. In a first step, one tries to find a "reference" (so-called) "equilibrium" state in which all hydrodynamical quantities are time-independent. In the simplest cases, it is also assumed that the equilibrium is truly static, i.e. $\boldsymbol{u} = 0$ *. The problem consists of finding a magnetic field and a pressure field satisfying the mechanical equilibrium condition derived from (I.2),

$$\nabla P = -\frac{1}{4\pi}\boldsymbol{B} \wedge (\nabla \wedge \boldsymbol{B}). \tag{4.1}$$

This problem will be discussed in chapter 8. Its solution leads to a variety of equilibrium configurations (e.g. tokamak, stellarator, spheromak, ...), most of which have the well-known toroidal symmetry.

(b) *Stability theory*. When an equilibrium solution is found, its stability against small perturbations must be studied. It so happens that no universally stable magnetic field confinement exists: the plasmas are plagued by an enormous variety of instabilities. Some of these can be explained in the framework of ideal MHD; for others, the role of dissipative processes is essential; finally some ("micro-instabilities") are produced by specific forms of the non-equilibrium particle distribution functions.

The identification of the instabilities, the calculation of their thresholds and growth rates, the evolution of an initially growing perturbation and its ultimate saturation, which may involve a complete reorganization (bifurcation) of the plasma state, the possible onset of a turbulent state, are some of the key problems in linear and non-linear stability theory.

* This assumption raises a problem. If $u = 0$, then $R_m = 0$ (see eq. 3.1) and ideal MHD is completely invalid! However, the only equation which remains non-trivial in table 3.1 is the momentum balance equation (I.2), which is *not* affected by the ideal MHD approximation. Nevertheless, the validity of studying small-velocity perturbations in stage (b) by using the ideal MHD model must be carefully questioned!

(c) *Transport theory*. As we know from the previous chapters, transport theory necessarily goes beyond the ideal hydrodynamical framework. The careful study of the dissipative processes is essential for the understanding of the leakage through the confining magnetic surfaces.

Here we are faced with a new problem. Contrary to our assumptions of the classical theory, the basic confining magnetic field is fundamentally inhomogeneous and curved. As a result, the motion of charged particles in such a field can be qualitatively (topologically) different from the motion in a straight, almost homogeneous field. As a result, *the transport theory must be very deeply reconsidered when dealing with a magnetically confined plasma*. This is a relatively new discovery (1967) which led to a new corpus of knowledge called the *Neoclassical transport theory*. The very unusual phenomena induced by the magnetic field geometry on the mechanisms of transport will be the object of the second volume of our work.

(d) *Heating theory*. Once a (not too unstable) "equilibrium" state is established, the temperature must be raised in order to reach internal ion temperatures of the order of 100 keV. Various means are at our disposal for achieving this goal. Ohmic (or better: Joule) heating by the electric current induced in a confined plasma is necessarily limited in scope. Indeed, we know from eq. (5.5.12) that the electrical conductivity $\sigma_{\parallel}$ grows like $T^{3/2}$ with temperature; thus, the higher the temperature, the less efficient becomes the Joule heating mechanism. One therefore uses *additional heating* procedures, such as neutral beam injection or RF heating.

In the first case, intense *highly energetic beams of neutral hydrogen or deuterium particles* are injected into the plasma; their neutrality ensures an easy penetration into the plasma. Ultimately, their energy is transferred to the plasma by collisions.

In the second method, *strong electromagnetic radiation beams* are injected into the plasma. Their frequency must be carefully chosen in order to ensure their penetration into the medium, but also the exchange of energy with the particles through resonances at various oscillation eigenmodes of the plasma, such as the *electron cyclotron waves (ECR), the ion cyclotron waves (ICR), the lower hybrid waves (LHR) or the Alfvén waves*. The interaction of the heating beam with the plasma produces strong deformations of the distribution function at certain well-defined positions in space and in certain regions of velocity space, determined by the resonance conditions. The resulting non-equilibrium distributions may strongly affect the stability properties, as well as the transport properties of the plasma.

This remark shows that *the four stages*, which we described separately, *are actually strongly interconnected*. Other important examples are easily found. For instance, it may well be important to include a non-vanishing velocity into the basic "equilibria" studied in stage (a). This is especially desirable in the

presence of heating by neutral beam injection. In this case, the effect of the viscosity can no longer be neglected and the complete equation (2.1) must be solved. Such studies are only in a beginning stage (see, e.g. Wobig 1985).

Another, very important example is the following. Certain types of instability studied in stage (b) (e.g. the tearing modes) may lead, through the so-called *magnetic reconnection* phenomenon, to a radical change of topology of the ideal magnetic surfaces determined in stage (a). Typical examples of such changes are the formation of magnetic islands or even the complete chaotization of the magnetic field. The reader is advised to have a mere look at the pictures represented in the paper by White (1984), in order to grasp the complexity of the structure of the magnetic field in a (more or less) realistic situation. Such new magnetic field topologies lead to a complete change of character of the transport mechanisms, giving rise to the so-called *anomalous transport*. These problems will however not be faced in the present volume.

The fusion theory community begins to be conscious of the necessity of an *integrated study* of all these aspects of the magnetic confinement. The treatment of these problems involves enormous difficulties, both in analysis and in numerical computations; extremely important results are, however, expected to come out in the future from such a global approach.

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Appendix

G1

Basis functions in velocity space

G1.1. Expansions around the reference distribution function

The transport theory of plasmas (or, for that matter, of any system) consists of the evaluation of certain macroscopic quantities of hydrodynamical interest, such as the fluxes of matter, momentum or energy. These quantities are characteristic of a system in a non-equilibrium state, hence they require the determination of the *non-equilibrium distribution function*. This is, in general, a very difficult problem.

All the plasma regimes studied in this book have one property in common. For varied reasons (discussed in the main text), their state is not very far from the equilibrium state. The distribution functions of the particles of species α can therefore be represented in the form

$$f^{\alpha}(\boldsymbol{v};\, t) = f^{\alpha 0}(\boldsymbol{v})[1 + \chi^{\alpha}(\boldsymbol{v})], \tag{1.1}$$

where $f^{\alpha 0}(\boldsymbol{v})$ is the *local plasma equilibrium distribution function*, defined in eq. (4.2.6), and $\chi^{\alpha}(\boldsymbol{v})$ is the deviation characterizing the non-equilibrium state. (We do not write down explicitly the dependence on either the spatial variable $\boldsymbol{x}$, or the time t, which are irrelevant in the present context.) It is assumed that

$$|\chi^{\alpha}(\boldsymbol{v})| \ll 1 \tag{1.2}$$

for all values of the variables. If we use the dimensionless velocity variables $\boldsymbol{c}$ (defined in eq. 4.3.4), we considerably simplify the notations (see eq. 4.3.8) and we are left with a definite mathematical problem. We want to construct various useful representations of functions of the form

$$f(\boldsymbol{c}) = \phi^{0}(c)[1 + \chi(\boldsymbol{c})]. \tag{1.3}$$

The natural thing to do for the study of the non-equilibrium deviation $\chi(\boldsymbol{c})$ is to expand along an infinite basis of polynomials spanning the functional space of $\chi(\boldsymbol{c})$. The definition of the function χ is then equivalent to the specification of the set of coefficients entering its expansion. This technique is

widely used in quantum mechanics. The choice of a "natural" or "useful" set of polynomials is dictated by the form of the reference function $\phi^0(c)$. In all the ultimate problems of kinetic theory, the distribution function enters as a weight function under an integral necessary for the calculation of average quantities. Hence, it is desirable that the polynomials of the functional basis be mutually orthogonal with respect to a scalar product, defined with $\phi^0(c)$ as a weight function.

This discussion is straightforward mathematics. Some complications arise, however, from the fact that we are dealing here with functions $\chi(\boldsymbol{c})$ of *three variables* (rather than a single one, as in elementary quantum mechanics). In particular, the expansions take quite different (though interrelated) forms according to the type of coordinates used (e.g. Cartesian or spherical coordinates). It appears useful to give here a brief, but comprehensive discussion of these questions, which are not often presented systematically either in mathematical books on special functions, or in books on kinetic theory.

G1.2. Reducible tensorial Hermite polynomials

It is well known from elementary quantum mechanics that the orthogonal basis associated with a Gaussian weight function is made up of the Hermite polynomials. These require a generalization, because we are interested in functions of three variables.

Specifically, let $\boldsymbol{c}$ be a (dimensionless) vector in three-dimensional Euclidean space. Its Cartesian components are denoted by c_r ($r = 1, 2, 3$) and its length is $c \equiv (c_1^2 + c_2^2 + c_3^2)^{1/2}$. The reference (or weight) function is

$$\phi^0(c) = (2\pi)^{-3/2}\, \mathrm{e}^{-c^2/2}, \tag{2.1}$$

which is normalized to one:

$$\int \mathrm{d}\boldsymbol{c}\, \phi^0(c) \equiv \int_{-\infty}^{\infty} \mathrm{d}c_1 \int_{-\infty}^{\infty} \mathrm{d}c_2 \int_{-\infty}^{\infty} \mathrm{d}c_3\, \phi^0(c) = 1. \tag{2.2}$$

The *reducible tensorial Hermite polynomials* (Grad 1949) are defined by a natural generalization of the Hermite polynomials in one dimension [Gradshtein and Ryzhik 1980, Abramowitz and Stegun 1965):

$$\tilde{H}^{(m)}_{r_1 \cdots r_m}(\boldsymbol{c}) = (-1)^m\, \mathrm{e}^{c^2/2} \frac{\partial}{\partial c_{r_1}} \frac{\partial}{\partial c_{r_2}} \cdots \frac{\partial}{\partial c_{r_m}} \mathrm{e}^{-c^2/2}. \tag{2.3}$$

Note that this object is a *tensor* of rank m (it has exactly m indices: $r_1, \ldots, r_m$). By construction, it is completely symmetric in all its indices.

Table 2.1
Reducible tensorial Hermite polynomials.

$$\tilde{H}^{(0)}(c) = 1$$
$$\tilde{H}^{(1)}_m(c) = c_m$$
$$\tilde{H}^{(2)}_{mn}(c) = c_m c_n - \delta_{mn}$$
$$\tilde{H}^{(3)}_{mnr}(c) = c_m c_n c_r - (c_m \delta_{nr} + c_n \delta_{mr} + c_r \delta_{mn})$$
$$\tilde{H}^{(4)}_{mnrs}(c) = c_m c_n c_r c_s - (c_m c_n \delta_{rs} + c_m c_r \delta_{ns} + c_m c_s \delta_{nr} + c_n c_r \delta_{ms} + c_n c_s \delta_{mr} + c_r c_s \delta_{mn}) + \delta_{mn}\delta_{rs} + \delta_{mr}\delta_{ns} + \delta_{ms}\delta_{nr}$$

The important property of the tensorial Hermite polynomials is their mutual orthogonality over the domain $(-\infty, \infty)$ of each variable c_r, and with $\phi^0(c)$ as a weight function:

$$(2\pi)^{-3/2} \int \mathrm{d}\boldsymbol{c}\, e^{-c^2/2}\, \tilde{H}^{(m)}_{r_1,\ldots,r_m}(\boldsymbol{c})\, \tilde{H}^{(n)}_{s_1,\ldots,s_n}(\boldsymbol{c})$$

$$= \delta_{mn} \underbrace{\left(\delta_{r_1 s_1}\delta_{r_2 s_2} \ldots \delta_{r_m s_m} + \delta_{r_1 s_2}\delta_{r_2 s_1} \ldots \delta_{r_m s_m} + \cdots\right)}_{\substack{m! \text{ terms corresponding to all permutations} \\ \text{of the set } (s_1, \ldots, s_m)}} \tag{2.4}$$

Thus, explicitly,

$$\int \mathrm{d}\boldsymbol{c}\, \phi^0(c)\, \tilde{H}^{(0)}\tilde{H}^{(0)} = 1,$$

$$\int \mathrm{d}\boldsymbol{c}\, \phi^0(c)\, \tilde{H}^{(1)}_r \tilde{H}^{(1)}_s = \delta_{rs},$$

$$\int \mathrm{d}\boldsymbol{c}\, \phi^0(c)\, \tilde{H}^{(2)}_{r_1 r_2} \tilde{H}^{(2)}_{s_1 s_2} = \delta_{r_1 s_1}\delta_{r_2 s_2} + \delta_{r_1 s_2}\delta_{r_2 s_1}. \tag{2.5}$$

The first few of these polynomials are listed in table 2.1.

The function $\chi(\boldsymbol{c})$ can be expanded as

$$\chi(\boldsymbol{c}) = \sum_{m=0}^{\infty} \frac{1}{m!} \tilde{h}^{(m)}_{r_1 \ldots r_m} \tilde{H}^{(m)}_{r_1 \ldots r_m}(\boldsymbol{c}). \tag{2.6}$$

(Summation over repeated tensor indices is understood, as usual.) Because of the orthogonality of the polynomials, the *reducible Hermitian moments* $\tilde{h}^{(m)}_{r_1 \dots r_m}$ are obtained as

$$\tilde{h}^{(m)}_{r_1 \dots r_m} = \int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ \chi(\boldsymbol{c})\ \tilde{H}^{(m)}_{r_1 \dots r_m}(\boldsymbol{c}). \tag{2.7}$$

Although these tensorial Hermite polynomials are used in several classical works [notably those by Grad (1949), who initiated the celebrated "moment method"], they present a serious disadvantage. Indeed, *there is no one-to-one correspondence between the reducible Hermitian tensors and the physical fluxes.* This is because each completely symmetric tensor $\tilde{H}^{(m)}_{r_1 \dots r_m}$ can be invariantly decomposed into combinations involving lower rank irreducible tensors. In simple words, any symmetric tensor $\boldsymbol{A}$ can be "*pruned*", by subtracting systematically the tensors obtained by *contraction* over all the available pairs of indices. Thus, for instance,

$$\tilde{A}_{rs} = A_{rs} + \left(\tfrac{1}{3}\tilde{A}_{mm}\right)\delta_{rs},$$

$$\tilde{A}_{rst} = A_{rst} + \tfrac{1}{5}\left(\tilde{A}_{rmm}\delta_{st} + \tilde{A}_{msm}\delta_{rt} + \tilde{A}_{mmt}\delta_{rs}\right). \tag{2.8}$$

The *irreducible tensor* A_{rs} has zero trace; similarly, the contraction of the *irreducible* tensor A_{rst} over any pair of indices yields zero.

As a result of this situation, if we wish to pick up, say, all the vectors in expansion (2.6), we have to look for them in all the odd-rank reducible tensors. It is, however, desirable to write the expansion of the function $\chi(\boldsymbol{c})$ in such a way as to group together all the terms corresponding to the same type of irreducible *anisotropy* (such as: vectors, traceless second-rank tensors, ...).

This can be achieved in two ways. One consists of performing the "pruning", as illustrated in (2.8), on all the reducible Hermite tensors. The result is the definition of a set of *irreducible tensorial Hermite polynomials*, which possess the desired property. This direct procedure easily yields the first few polynomials, but is not simply amenable to a general construction. A second method consists of going through an intermediate change of variables, which is of interest in itself.

G1.3. Spherical harmonics, Laguerre–Sonine polynomials, Burnett functions

We go over from the Cartesian coordinates (c_1, c_2, c_3) to the *spherical coordinates* (x, θ, φ) by defining the variable x as

$$\tfrac{1}{2}c^2 = x, \quad \text{or} \quad c = \sqrt{2x} \tag{3.1}$$

(physically, x represents the dimensionless kinetic energy). The individual components transform as

$$c_1 = \sqrt{2x}\sin\theta\cos\varphi,$$

$$c_2 = \sqrt{2x}\sin\theta\sin\varphi,$$

$$c_3 = \sqrt{2x}\cos\theta. \tag{3.2}$$

We note that the Jacobian of this transformation is

$$J = \sqrt{2x}\sin\theta. \tag{3.3}$$

The *reference function* now depends on the single variable x. We thus define

$$\phi^0(x) = \frac{2}{\sqrt{\pi}}\mathrm{e}^{-x} \tag{3.4}$$

which is normalized as

$$\int_0^\infty \mathrm{d}x\sqrt{x}\,\phi^0(x) = 1. \tag{3.5}$$

We define a natural *scalar product* of two (complex) functions F, G as

$$\langle F\,|\,G\rangle = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{d}x\sqrt{x}\,\mathrm{e}^{-x}\frac{1}{4\pi}\int_0^\pi \mathrm{d}\theta\,\sin\theta\int_0^{2\pi}\mathrm{d}\varphi\,F^*(x,\theta,\varphi)$$

$$\times G(x,\theta,\varphi). \tag{3.6}$$

(This definition clearly yields: $\langle 1\,|\,1\rangle = 1$.)

With this scalar product, the natural basis for the expansion of any function of the *angles* θ, φ is provided by the well-known *spherical harmonics* $Y_l^m(\theta,\varphi)$, defined as [Margenau and Murphy (1943), Morse and Feshbach (1953), Korn and Korn (1968), Smirnov (1972), or any good book on quantum mechanics or electromagnetism]

$$Y_l^m(\theta,\varphi) = (-1)^m a_l^m P_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\varphi}, \tag{3.7}$$

where $P_l^m(z)$ is the associated Legendre function; l is a non-negative integer,

Table 3.1
Spherical harmonics.

$Y_0^0 = 1$

$Y_1^0 = \sqrt{3}\cos\theta$

$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{2}}\sin\theta\, e^{\pm i\varphi}$

$Y_2^0 = \sqrt{\frac{5}{4}}(3\cos^2\theta - 1)$

$Y_2^{\pm 1} = \mp\sqrt{\frac{15}{2}}\cos\theta\sin\theta\, e^{\pm i\varphi}$

$Y_2^{\pm 2} = \sqrt{\frac{15}{8}}\sin^2\theta\, e^{\pm 2i\varphi}$

whereas m is a positive or negative integer (or zero) in the range $|m| \leqslant l$. The normalization coefficient is

$$a_l^m = \left((2l+1)\frac{(l-m)!}{(l+m)!}\right)^{1/2}. \tag{3.8}$$

The spherical harmonics have the following orthogonality property:

$$\frac{1}{4\pi}\int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\varphi\, Y_{l'}^{m'*}(\theta, \varphi)\, Y_l^m(\theta, \varphi) = \delta_{l,l'}\delta_{m,m'}. \tag{3.9}$$

The first few spherical harmonics are listed in table 3.1.

We now complete the basis in order to take into account the energy variable x. With the weight function (3.4), the natural basis functions are related to the *associated Laguerre polynomials*, $L_n^{l+1/2}(x)$, (Margenau and Murphy 1943, Gradshtein and Ryzhik 1980, Abramowitz and Stegun 1965, Smirnov 1972). These are frequently called *Sonine polynomials* in the kinetic theory literature. We therefore call them here *Laguerre–Sonine polynomials*. Their exact definition varies slightly from one author to another (the differences are in the sign of the odd polynomials and the normalization factors). We adopt here the following definition:

$$L_n^{l+1/2}(x) = \alpha_{nl}\sum_{m=0}^{n}(-)^{m+n}\frac{1}{2^{n-m}}\frac{n!}{m!(n-m)!}\frac{(2n+2l+1)!!}{(2m+2l+1)!!}\,x^m, \tag{3.10}$$

where l and n are non-negative integers (or zero). The normalization coefficient is

$$\alpha_{nl} = \left(\frac{2^{n+l}}{n!(2n+2l+1)!!} \right)^{1/2}. \tag{3.11}$$

A compact representation of these polynomials is

$$L_n^{l+1/2}(x) = (-)^n \alpha_{nl}\, \mathrm{e}^x\, x^{-(l+1/2)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (\mathrm{e}^{-x}\, x^{n+l+1/2}). \tag{3.12}$$

The first few Laguerre–Sonine polynomials are listed in table 3.2.

The Laguerre–Sonine polynomials are very useful for the expansion of functions $g(x)$, depending on the single variable x. These functions are represented as

$$g(x) = \sum_{n=0}^{\infty} \lambda_n^{[l+1/2]} L_n^{l+1/2}(x), \tag{3.13}$$

where the coefficients are given by

$$\lambda_n^{[l+1/2]} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \mathrm{d}x\, \mathrm{e}^{-x}\, x^{(l+1)/2} L_n^{l+1/2}(x)\, g(x). \tag{3.14}$$

The choice of the particular family of Laguerre–Sonine polynomials (i.e. the choice of the value of l) appropriate to the problem is dictated by the context.

The spherical harmonics and the Laguerre–Sonine polynomials are now combined in the construction of a set of basis functions to be used for the expansion of any function of the three variables x, θ, φ;

$$B_{nl}^m(x, \theta, \varphi) = x^{l/2} L_n^{l+1/2}(x)\, Y_l^m(\theta, \varphi). \tag{3.15}$$

These functions are properly called *Burnett functions*, because they were extensively used by Burnett (1935) in his classical work on the Boltzmann equation. An extensive study of their properties can be found in the papers by Weinert (1981, 1982). Combining definition (3.6) and results (3.9) and (3.14), we find

$$\langle B_{n'l'}^{m'} | B_{nl}^m \rangle = \delta_{nn'} \delta_{ll'} \delta_{mm'}. \tag{3.16}$$

The expansion of an arbitrary function $\chi(x, \theta, \varphi)$ on the Burnett basis is performed as

$$\chi(x, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{nl}^m B_{nl}^m(x, \theta, \varphi). \tag{3.17}$$

Table 3.2
Laguerre–Sonine polynomials.

$l = 0$

$$L_0^{1/2}(x) = 1$$

$$L_1^{1/2}(x) = \sqrt{\tfrac{2}{3}}\,(x - \tfrac{3}{2})$$

$$L_2^{1/2}(x) = \sqrt{\tfrac{2}{15}}\,(x^2 - 5x + \tfrac{15}{4})$$

$$L_3^{1/2}(x) = \frac{2}{3\sqrt{35}}(x^3 - \tfrac{21}{2}x^2 + \tfrac{105}{4}x - \tfrac{105}{8})$$

$$L_4^{1/2}(x) = \frac{\sqrt{2}}{9\sqrt{35}}(x^4 - 18x^3 + \tfrac{189}{2}x^2 - \tfrac{315}{2}x + \tfrac{945}{16})$$

$l = 1$

$$L_0^{3/2}(x) = \sqrt{\tfrac{2}{3}}$$

$$L_1^{3/2}(x) = \frac{2}{\sqrt{15}}(x - \tfrac{5}{2})$$

$$L_2^{3/2}(x) = \frac{2}{\sqrt{105}}(x^2 - 7x + \tfrac{35}{4})$$

$$L_3^{3/2}(x) = \frac{2\sqrt{2}}{9\sqrt{35}}(x^3 - \tfrac{27}{2}x^2 + \tfrac{189}{4}x - \tfrac{315}{8})$$

$$L_4^{3/2}(x) = \frac{3}{9\sqrt{385}}(x^4 - 22x^3 + \tfrac{297}{2}x^2 - \tfrac{693}{2}x + \tfrac{3465}{16})$$

$l = 2$

$$L_0^{5/2}(x) = \frac{2}{\sqrt{15}}$$

$$L_1^{5/2}(x) = \frac{2\sqrt{2}}{\sqrt{105}}(x - \tfrac{7}{2})$$

$$L_2^{5/2}(x) = \frac{2\sqrt{2}}{3\sqrt{105}}(x^2 - 9x + \tfrac{63}{4})$$

$$L_3^{5/2}(x) = \frac{4}{9\sqrt{385}}(x^3 - \tfrac{33}{2}x^2 + \tfrac{297}{4}x - \tfrac{693}{8})$$

$$L_4^{5/2}(x) = \frac{2\sqrt{2}}{9\sqrt{5005}}(x^4 - 26x^3 + \tfrac{429}{2}x^2 - \tfrac{1287}{2}x + \tfrac{9009}{16})$$

From the orthogonality property of the Burnett functions follows the definition of the expansion coefficients,

$$b_{nl}^m = \frac{2}{4\pi\sqrt{\pi}} \int_0^\infty \mathrm{d}x \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \, \sin\theta \sqrt{x}\, \mathrm{e}^{-x}\, B_{nl}^{m*}(x, \theta, \varphi)\, \chi(x, \theta, \varphi). \tag{3.18}$$

These coefficients will be called *Burnett moments*.

Expansion (3.16) can be written in the following, more explicit form, using (3.16),

$$\chi(x, \theta, \varphi) = \sum_{n=0}^{\infty} b^0_{2n,0} L_n^{1/2}(x) + \sum_{n=0}^{\infty} \sum_{m=-1}^{1} b^m_{2m+1,1} L_n^{3/2}(x) \sqrt{x}\, Y_1^m(x)$$

$$+ \sum_{n=0}^{\infty} \sum_{m=-2}^{2} b^m_{2n+2,2} L_n^{5/2}(x)\, x Y_2^m(\theta, \varphi) + \cdots . \qquad (3.19)$$

This form realizes the purpose announced at the end of section G1.2. Indeed, all the terms in the first group ($l = 0$) depend only on the length of the vector, c (i.e. on x); hence the moments $b_{2n,0}$ are *scalars*.

In the second group ($l = 1$) we find, for given n, the combination

$$\left(b^{-1}_{2n+1,1}\sqrt{x} \sin\theta\, e^{-i\varphi} + b^1_{2n+1,1}\sqrt{x} \sin\theta\, e^{i\varphi} + b^0_{2n+1,1}\sqrt{x} \cos\theta\right).$$

Because of (3.2), it is easily recognized that this combination is proportional to the scalar product of the vector $\boldsymbol{c}$ with a vector $\boldsymbol{b}_{2n+1,1}$. In other words, *the three moments* $b^m_{2n+1,2}$ *for* $m = -1, 0, 1$, *are the three components of a vector.*

A similar argument leads to the conclusion that the five components $xY_2^m(\theta, \varphi)$ correspond to the components of the tensor $(\boldsymbol{cc} - \frac{1}{3}c^2 \boldsymbol{I})$. Hence, the five moments $b^m_{2n+2,2}$ ($m = -2, -1, 0, 1, 2$) are the components of a *second-rank tensor with zero trace.*

In conclusion, each group of terms in (3.19), corresponding to a given value of l, contains only moments of the same tensorial character.

G1.4. Irreducible tensorial Hermite polynomials

Although the expansions of the form (3.19) explicitly display the various anisotropies and are widely used in the kinetic theory literature, they have a major disadvantage. Because of the use of spherical coordinates, the notations become quite "distorted" and untransparent. The use of Cartesian coordinates is much to be preferred for the expansions. We thus revert to the coordinates c_r ($r = 1, 2, 3$) used in section G1.2.

Consider the first group of terms ($l = 0$) in (3.19). The polynomials $L_n^{1/2}(x)$, of degree n in x, become polynomials of degree $2n$ in c (the length of the vector $\boldsymbol{c}$); they will be denoted by $H^{(2n)}(\boldsymbol{c})$. This group of terms will then be rewritten as

$$\sum_{n=0}^{\infty} b^0_{2n,0} L_n^{1/2}(x) \to \sum_{n=0}^{\infty} h^{(2n)} H^{(2n)}(\boldsymbol{c}). \qquad (4.1)$$

The polynomials $H^{(2n)}(\boldsymbol{c})$ will be called *scalar (irreducible) Hermite polynomials*. Their exact definition will be given below.

Consider now the second group of terms ($l=1$) in (3.19). As explained above, the three functions $\sqrt{x}\, Y_1^m(\theta, \varphi)$ ($m=-1, 0, 1$) correspond to the three components of the *vector* $\boldsymbol{c}$. Hence, these terms may be written in the form

$$\sum_{n=0}^{\infty} \sum_{m=-1}^{1} b_{2n+1,1}^{m} L_n^{3/2}(x)\, x^{1/2} Y_1^m(\theta, \varphi) \to \sum_{n=0}^{\infty} h_r^{(2n+1)} c_r \hat{H}_{[2n]}(c)$$

$$\equiv \sum_{n=0}^{\infty} h_r^{(2n+1)} H_r^{(2n+1)}(\boldsymbol{c}) \quad (4.2)$$

with the usual summation over the repeated vector indices r. $\hat{H}_{[2n]}$ is clearly a polynomial of degree $2n$ in variable c. The moments $h_r^{(2n+1)}$ ($r=1, 2, 3$), for each value of n, are the components of a *vector*. A similar transformation is performed on the other terms of (3.19), which is rewritten as

$$\chi(\boldsymbol{c}) = \sum_{n=0}^{\infty} h^{(2n)} H^{(2n)}(\boldsymbol{c}) + \sum_{n=0}^{\infty} h_r^{(2n+1)} H_r^{(2n+1)}(\boldsymbol{c})$$

$$+ \sum_{n=1}^{\infty} h_{rs}^{(2n)} H_{rs}^{(2n)}(\boldsymbol{c}) + \cdots . \quad (4.3)$$

The coefficients of this expansion are called *irreducible Hermitian moments*, and are qualified by their tensorial nature. Thus, $H^{(2n)}$ are *scalar Hermitian moments*, $h_r^{(2n+1)}$ are *vectorial Hermitian moments*, $h_{rs}^{(2n)}$ are *traceless tensorial Hermitian moments*, etc. It is shown in the main text that these irreducible Hermitian moments have very interesting properties, which make them supremely suitable for transport theory.

The basis functions $H^{(2n)}(\boldsymbol{c})$, $H_r^{(2n+1)}(\boldsymbol{c})$, etc. are also characterized by their irreducible tensorial character. Their form is obtained, up to an appropriate normalization constant, from the equivalences expressed in (4.1), (4.2), together with relations (3.1), (3.2). We thus obtain the following relations between the first three classes of irreducible tensorial Hermite polynomials and the Laguerre–Sonine polynomials,

$$H^{(2n)}(\boldsymbol{c}) = L_n^{1/2}\left(\tfrac{1}{2}c^2\right),$$

$$H_r^{(2n+1)}(\boldsymbol{c}) = \sqrt{\tfrac{3}{2}}\, c_r L_n^{3/2}\left(\tfrac{1}{2}c^2\right),$$

$$H_{rs}^{(2n)}(\boldsymbol{c}) = \sqrt{\tfrac{15}{8}}\left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right) L_{n-1}^{5/2}\left(\tfrac{1}{2}c^2\right). \quad (4.4)$$

Table 4.1
Irreducible tensorial Hermite polynomials

Scalars

$$H^{(0)}(\boldsymbol{c}) = 1$$

$$H^{(2)}(\boldsymbol{c}) = \frac{1}{\sqrt{6}}(c^2 - 3)$$

$$H^{(4)}(\boldsymbol{c}) = \frac{1}{2\sqrt{30}}(c^4 - 10c^2 + 15)$$

$$H^{(6)}(\boldsymbol{c}) = \frac{1}{12\sqrt{35}}(c^6 - 21c^4 + 105c^2 - 105)$$

Vectors

$$H_r^{(1)}(\boldsymbol{c}) = c_r$$

$$H_r^{(3)}(\boldsymbol{c}) = \frac{1}{\sqrt{10}}c_r(c^2 - 5)$$

$$H_r^{(5)}(\boldsymbol{c}) = \frac{1}{2\sqrt{70}}c_r(c^4 - 14c^2 + 35)$$

$$H_r^{(7)}(\boldsymbol{c}) = \frac{1}{12\sqrt{105}}c_r(c^6 - 27c^4 + 189c^2 - 315)$$

Traceless tensors of second rank

$$H_{rs}^{(2)}(\boldsymbol{c}) = \frac{1}{\sqrt{2}}(c_r c_s - \tfrac{1}{3}c^2\delta_{rs})$$

$$H_{rs}^{(4)}(\boldsymbol{c}) = \frac{1}{2\sqrt{7}}(c_r c_s - \tfrac{1}{3}c^2\delta_{rs})(c^2 - 7)$$

$$H_{rs}^{(6)}(\boldsymbol{c}) = \frac{1}{12\sqrt{7}}(c_r c_s - \tfrac{1}{3}c^2\delta_{rs})(c^4 - 18c^2 + 63)$$

Third-order anisotropy

$$H_{rsp}^{(3)}(\boldsymbol{c}) = c_r c_s c_p - \tfrac{1}{5}c^2(c_r\delta_{sp} + c_s\delta_{rp} + c_p\delta_{rs})$$

Fourth-order anisotropy

$$H_{rspq}^{(4)}(\boldsymbol{c}) = c_r c_s c_p c_q - \tfrac{1}{7}c^2(c_r c_s\delta_{pq} + c_r c_p\delta_{sq} + c_r c_q\delta_{sp} + c_s c_p\delta_{rq} + c_s c_q\delta_{rp} + c_p c_q\delta_{rs}) + \tfrac{1}{35}c^4(\delta_{rs}\delta_{pq} + \delta_{rp}\delta_{sq} + \delta_{rq}\delta_{sp})$$

Their explicit form is obtained from these formulae, combined with (3.10), (3.11).

Scalar Hermite polynomials

$$H^{(2n)}(\boldsymbol{c}) = \beta_{2n}^{[s]} \sum_{m=0}^{n} (-1)^{m+n} \frac{n!}{m!(n-m)!} \frac{(2n+1)!!}{(2m+1)!!} c^{2m},$$

$$\beta_{2n}^{[s]} = \left(\frac{1}{2^n n!(2n+1)!!} \right)^{1/2}. \tag{4.5}$$

Vector Hermite polynomials

$$H_r^{(2n+1)}(\boldsymbol{c}) = \beta_{2n+1}^{[v]} c_r \sum_{m=0}^{n} (-1)^{m+n} \frac{n!}{m!(n-m)!} \frac{(2n+3)!!}{(2m+3)!!} c^{2m},$$

$$\beta_{2n+1}^{[v]} = \left(\frac{3}{2^n n!(2n+3)!!} \right)^{1/2}. \tag{4.6}$$

Traceless tensor Hermite polynomials

$$H_{rs}^{(2n)}(\boldsymbol{c}) = \beta_{2n}^{[T]} \left(c_r c_s - \tfrac{1}{3} c^2 \delta_{rs} \right) \times \sum_{m=0}^{n-1} (-1)^{m+n-1} \frac{(n-1)!}{m!(n-m-1)!} \frac{(2n+3)!!}{(2m+5)!!} c^{2m},$$

$$\beta_{2n}^{[T]} = \left(\frac{15}{2^n (n-1)!(2n+3)!!} \right)^{1/2}. \tag{4.7}$$

A list of the first polynomials in each of these families is given in table 4.1. It is easily checked that these polynomials are identical to those obtained from the reducible Hermite polynomials by the "pruning" procedure described at the end of section G1.2. The relation between reducible and irreducible Hermite polynomials is the following:

$$\tilde{H}^{(0)} = H^{(0)}, \qquad \tilde{H}_r^{(1)} = H_r^{(1)},$$

$$\tilde{H}_{rs}^{(2)} = \sqrt{2} \left(H_{rs}^{(2)} + (\sqrt{3})^{-1} H^{(2)} \delta_{rs} \right),$$

$$\tilde{H}_{rsp}^{(3)} = H_{rsp}^{(3)} + \sqrt{\tfrac{2}{5}} \left(H_r^{(3)} \delta_{sp} + H_s^{(3)} \delta_{rp} + H_p^{(3)} \delta_{rs} \right). \tag{4.8}$$

The irreducible Hermite polynomials have the following *orthogonality properties*, associated with the Maxwellian weight function $\phi^0(c)$, defined in (2.1),

Scalar Hermite polynomials:

$$\int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ H^{(2m)}(\boldsymbol{c})\ H^{(2n)}(\boldsymbol{c}) = \delta_{mn}. \tag{4.9}$$

Vector Hermite polynomials:

$$\int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ H_r^{(2m+1)}(\boldsymbol{c})\ H_s^{(2n+1)}(\boldsymbol{c}) = \delta_{mn}\delta_{rs}. \tag{4.10}$$

Traceless tensor Hermite polynomials:

$$\int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ H_{rs}^{(2m)}(\boldsymbol{c})\ H_{pq}^{(2n)}(\boldsymbol{c}) = \tfrac{1}{2}\delta_{mn}\left(\delta_{rp}\delta_{sq} + \delta_{rq}\delta_{sp} - \tfrac{2}{3}\delta_{rs}\delta_{pq}\right). \tag{4.11}$$

From this orthogonality property follows the definition of the *irreducible Hermitian moments*, i.e. the coefficients of expansion (4.3),

$$h^{(p)}_{r_1\ldots r_q} = \int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ H^{(p)}_{r_1\ldots r_q}(\boldsymbol{c})\ \chi(\boldsymbol{c}), \tag{4.12}$$

where $H^{(p)}_{\ldots}(\boldsymbol{c})$ is any irreducible tensorial Hermite polynomial of degree p. (Note that, necessarily, such an object is a tensor of rank less than or equal to the degree p: $q \leqslant p$).

We now list a set of useful recurrence relations and of differential relations for the irreducible Hermite polynomials. The remarkable fact about these is that all but one of them involve at most three polynomials.

Direct product with $\boldsymbol{c}$:

$$c_r H^{(2n)} = \sqrt{\frac{2n+3}{3}}\ H_r^{(2n+1)} + \sqrt{\frac{2n}{3}}\ H_r^{(2n-1)},$$

$$c_r H_s^{(2n+1)} = \sqrt{\frac{2(2n+5)}{5}}\ H_{rs}^{(2n+2)} + \sqrt{\frac{4n}{5}}\ H_{rs}^{(2n)}$$

$$+ \tfrac{1}{3}\delta_{rs}\left(\sqrt{6(n+1)}\,H^{(2n+2)} + \sqrt{3(2n+3)}\,H^{(2n)}\right). \tag{4.13}$$

Contraction with $\boldsymbol{c}$:

$$c_r H_r^{(2n+1)} = \sqrt{6(n+1)}\, H^{(2n+2)} + \sqrt{3(2n+3)}\, H^{(2n)},$$

$$c_r H_{rs}^{(2n)} = \frac{2}{3}\left(\sqrt{5n}\, H_s^{(2n+1)} + \sqrt{\frac{5(2n+3)}{2}}\, H_s^{(2n-1)}\right). \tag{4.14}$$

Gradient:

$$\frac{\partial}{\partial c_r} H^{(2n)} = \sqrt{\frac{2n}{3}}\, H_r^{(2n-1)},$$

$$\frac{\partial}{\partial c_r} H_s^{(2n+1)} = \sqrt{\frac{4n}{5}}\, H_{rs}^{(2n)} + \sqrt{\frac{2n+3}{3}}\, H^{(2n)}\delta_{rs}. \tag{4.15}$$

Divergence:

$$\frac{\partial}{\partial c_r} H_r^{(2n+1)} = \sqrt{3(2n+3)}\, H^{(2n)},$$

$$\frac{\partial}{\partial c_r} H_{rs}^{(2n)} = \frac{10}{3}\sqrt{\frac{2n+3}{10}}\, H_s^{(2n-1)}. \tag{4.16}$$

Contraction with $\boldsymbol{c}$ *of the gradient*:

$$c_p \frac{\partial}{\partial c_p} H^{(2n)} = 2nH^{(2n)} + \sqrt{2n(2n+1)}\, H^{(2n-2)},$$

$$c_p \frac{\partial}{\partial c_p} H_r^{(2n+1)} = (2n+1) H_r^{(2n+1)} + \sqrt{2n(2n+3)}\, H_r^{(2n-1)},$$

$$c_p \frac{\partial}{\partial c_p} H_{rs}^{(2n)} = 2nH_{rs}^{(2n)} + \sqrt{(2n-2)(2n+3)}\, H_{rs}^{(2n-2)}. \tag{4.17}$$

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